Let $\mathcal{U} = \{\mathcal{U}_\alpha\}$ such that $X \subset \cup U^o_\alpha$.

Then $C^{\mathcal{U}}_n(X) = \{\sum r_i\sigma_i \mid \sigma_i \subset U_\alpha \text{ for some } \alpha\}$ is a subgroup of $C_n(X)$.

$\partial(C^{\mathcal{U}}_n(X)) \subset C^{\mathcal{U}}_{n-1}(X)$ and $\partial^2 = 0$. Thus $\exists H^{\mathcal{U}}_n(X)$

Prop 2.2.1: The inclusion map $i : C^{\mathcal{U}}_n(X) \to C_n(X)$ is a chain homotopy equivalence.

I.e., $\exists \rho : C_n(X) \to C^{\mathcal{U}}_n(X)$ wuthc that $i\rho$ and ρi are chain homotopic to the identity.

Hence i induces an isomorphism $H^{\mathcal{U}}_n(X) \cong H_n(X)$.

(1) Barycentric subdivision of of (ideal) simplices.

Simplex $[v_0, ..., v_n] = \{\sum t_iv_i \mid \sum t_i = 1, t_i \geq 0\}$

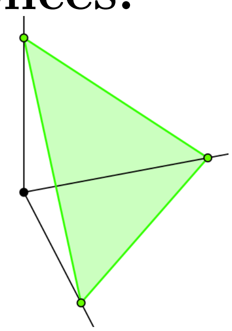


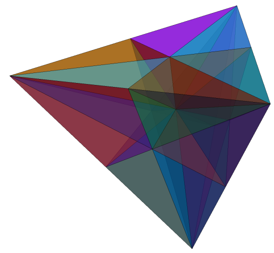
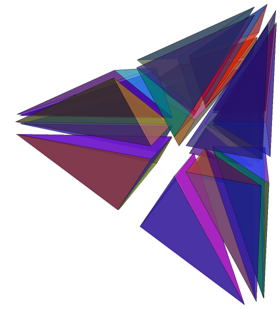
Figure 1: http://www.wikiwand.com/en/Simplex

The barycenter = center of gravity = $b = \sum\limits_{i=0}^{n} \frac{1}{n+1}v_i$

Barycentric subdivision: decompose $[v_0, ..., v_n]$ into the $n-$simplices $[b, w_0, ..., w_{n-1}]$, inductively.

Divide each edge $[v_1, v_2]$ in half, forming 2 new edges $[b, v_1]$, $[b, v_2]$.

Note: $diam[b, v_i] = ||v_i - b|| = \frac{1}{2}||v_2 - v_1|| = \frac{1}{2}(diam[v_1, v_2])$

http://drorbn.net/AcademicPensieve/2010-06/

Claim:
If b is a barycenter of $[v_0, ..., v_{k-1}]$, then $||b - v_i|| \leq \left(\frac{k-1}{k}\right) ||v_j - v_k||$.

Thus diam $[b, w_0, ..., w_{k-1}] \leq \left(\frac{k-1}{k}\right) diam[v_0, ..., v_n]$

Note: Claim is true for $k = 2$. Suppose claim is true for $k = n - 1$.

Suppose all the faces of $[v_0, ..., v_n]$ have been subdivided. For all $n-1$-simplices $[w_0, ..., w_{n-1}]$ in this subdivision, form the $n-$simplices $[b, w_0, ..., w_{n-1}]$, where b is the barycenter of $[v_0, ..., v_n]$

By induction $||w_i - w_j|| \leq \left(\frac{n-1}{n}\right) ||v_l - v_k||$.

Let b_i be the barycenter of $[v_0, ..., \widehat{v_i}, ..., v_n]$

$$b = \sum_{j=0}^{n} \frac{1}{n+1} v_j = \left(\frac{1}{n+1}\right) v_i + \sum_{j \neq i} \left(\frac{1}{n+1}\right) v_j = \left(\frac{1}{n+1}\right) v_i + \left(\frac{n}{n+1}\right) \sum_{j \neq i} \left(\frac{1}{n}\right) v_j$$

$$= \left(\frac{1}{n+1}\right) v_i + \left(\frac{n}{n+1}\right) b_i$$

Thus $||b - v_i|| = \left(\frac{n}{n+1}\right) ||b_i - v_i|| \leq \left(\frac{n}{n+1}\right) ||v_j - v_i||$

Thus $diam[b, w_0, ..., w_{n-1}] \leq \left(\frac{n}{n+1}\right) diam[v_0, ..., v_n]$

Thus repeated barycentric subdivision leads to simplices whose diameter is arbitrarily small.

2. Barycentric subdivision of Linear Chains

For Y convex, define $LC_n(Y) = \{\lambda : \Delta^n \to Y \mid \lambda \text{ is linear } \}$

$\partial(LC_n(Y)) \subset LC_{n-1}(Y)$.

For convenience, define $LC_{-1}(Y) = \mathbb{Z} =< [\emptyset] >$ where $\partial[v] = [\emptyset]$

If $b \in Y$, define homomorphism $b : LC_n(Y) \to LC_{n+1}(Y)$, $b([w_0, ..., w_n]) = [b, w_0, ..., w_n]$, the cone operator.

$\partial b([w_0, ..., w_n]) = \partial[b, w_0, ..., w_n] = [w_0, ..., w_n] - b\partial[w_0, ..., w_n]$.

Thus if $\alpha = \sum_{i=1}^{n} r_i\lambda_i$, then $(\partial \circ b)(\alpha) = \alpha - (b \circ \partial)(\alpha), \forall \alpha \in LC_n(Y)$.

$(\partial \circ b)(\alpha) + (b \circ \partial)(\alpha) = \alpha$

That is $\partial \circ b + b \circ \partial = id - 0$, where id = the identity homomorphism and 0 = the constant zero homomorphism on $LC_n(Y)$.

Thus b is a chain homotopy between the identity map and the zero homomorphism on the augmented chain complex $LC(Y)$.

Define subdivision homomorphism $S : LC_n(Y) \to LC_n(Y)$ by induction on n.

Let $\lambda : \Delta^n \to Y$ be a generator of $LC_n(Y)$.

Let $b_\lambda = \lambda(b)$ where b is the barycenter of Δ^n.

Define $S([\emptyset]) = [\emptyset]$ and $S(\lambda) = b_\lambda(S(\partial(\lambda)))$

Ex: If $\lambda = [v]$, then $b_\lambda = v$ and

$$S([v]) = b_\lambda(S(\partial([v]))) = v(S([\emptyset])) = v([\emptyset]) = [v].$$

Thus S is the identity on $LC_{-1}(Y)$ and $LC_0(Y)$.

Ex: If $\lambda = [v, w]$, $S([v, w]) = b_\lambda(S(\partial([v, w])))$

$$= b_\lambda(S([w]) - S([v])) = b_\lambda([w] - [v]) = [b_\lambda, w] - [b_\lambda, v].$$

Ex: If $\lambda = [u, v, w]$, $S(u, [v, w]) = b_\lambda(S(\partial([u, v, w])))$

$= b_\lambda(S([v, w]) - S([u, w]) + S([u, v]))$

$= b_\lambda([b_{v,w}, w] - [b_{v,w}, v] - ([b_{u,w}, w] - [b_{u,w}, u]) + [b_{u,v}, v] - [b_{u,v}, u])$

$= [b_\lambda, b_{v,w}, w] - [b_\lambda, b_{v,w}, v] - [b_\lambda, b_{u,w}, w] + [b_\lambda, b_{u,w}, u] + [b_\lambda, b_{u,v}, v] - [b_\lambda, b_{u,v}, u]$

If λ is an embedding, $S(\lambda)$ is the alternating sum of the simplices in the barycentric subdivision of λ.

Claim: S is a chain homotopy between $LC_n(Y)$ and itself.

That is $\partial S = S\partial$.

Proof by induction on n:

True for $n = -1, 0$ since $S = id$.

$\partial(S(\lambda)) = \partial(b_\lambda(S(\partial(\lambda)))) = (1 - b_\lambda\partial)(S(\partial(\lambda)))$

$$= S(\partial(\lambda)) - b_\lambda(\partial(S(\partial(\lambda)))) = S(\partial(\lambda)) - b_\lambda(S(\partial(\partial(\lambda))))$$

$$= S(\partial(\lambda)) - b_\lambda(S(0) = S(\partial(\lambda))$$

Define a chain homotopy between S and id,

$$T : LC_n(Y) \to LC_{n+1}(Y) \text{ inductively:}$$

$T = 0$ for $n = -1$, and $T(\lambda) = b_\lambda(\lambda T \partial \lambda)$.