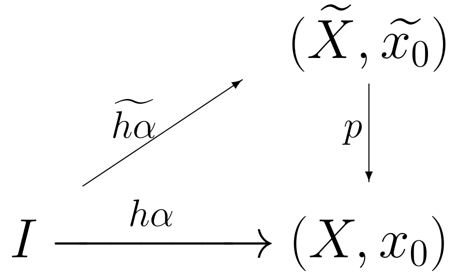
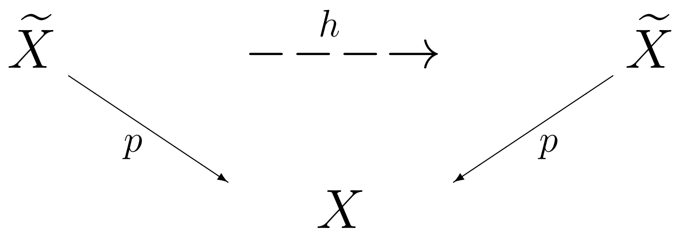
Prop. Let $H_0 = p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$
$\varphi : \pi_1(X, x_0) \to p^{-1}(x_0), \quad \varphi(\alpha) = \widetilde{\alpha}(1)$ induces

$$\Phi : \pi_1(X, x_0)/H_0 \to p^{-1}(x_0),$$

$$\Phi(H_0[\alpha]) = \widetilde{h\alpha}(1) = \widetilde{\alpha}(1)$$



If $\widetilde{X}$ path connected, then Φ is bijective.



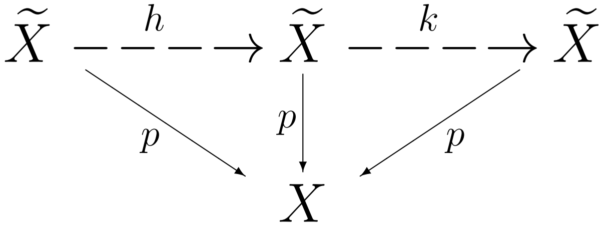
Definition 0.1.

$$\mathcal{C}(\widetilde{X}, p, X) = \{h : \widetilde{X} \to \widetilde{X} \mid h \text{ is a covering transformation for } p\}$$
$$= \{h : \widetilde{X} \to \widetilde{X} \mid h \text{ is a homeomorphism and } p = h \circ p\}$$

Examples:
$\mathcal{C}(\mathbb{R}, p = e^{2\pi i\theta}, S^1) = \{h_j : \mathbb{R} \to \mathbb{R}, h_j(x) = x + j \mid j \in \mathbb{Z}\} \simeq \mathbb{Z}$

$\mathcal{C}(S^1, p = z^k, S^1) =$
$\{h_j : S^1 \to S^1, h_j(e^x) = e^{x+2\pi i\frac{j}{k}} \mid j \in \{0, 1, ..., k-1\}\} \simeq \mathbb{Z}_k$

Note: $\mathcal{C}(\widetilde{X}, p, X)$ is a group under composition:



Definition 0.2. Suppose $H < G$. The normalizer of H in G is
$$N(H) = \{g \in G \mid gHg^{-1} = H\}$$

Note: $N(H) < G$ and $H \lhd N(H)$

$N(H)$ is the largest subgp of G containing H as a normal subgp.

Example 0.3. $N(\{e\}) = G$

Example 0.4. $N(H) = G$ if G is abelian.

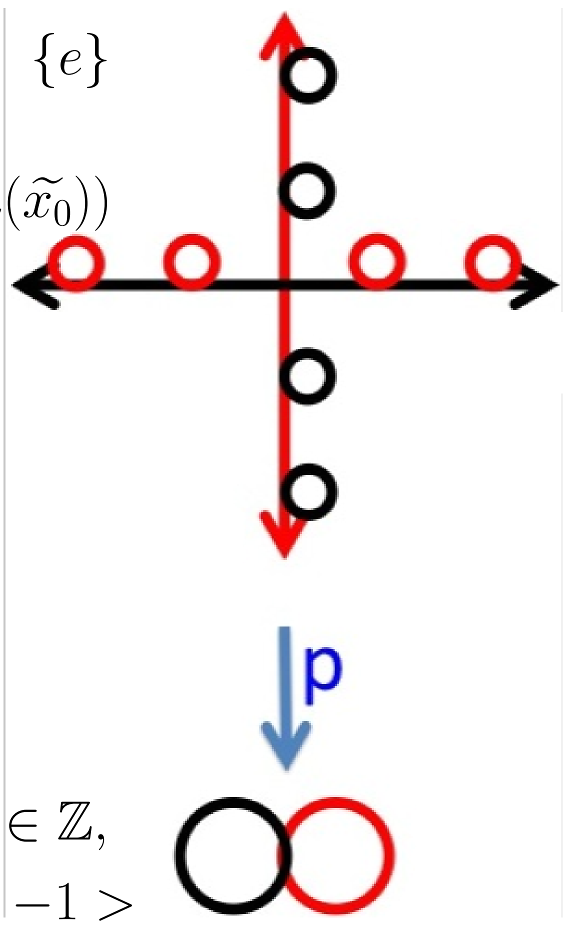
Claim: Let $H_0 = p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$. Then $\mathcal{C}(\widetilde{X}, p, X) \simeq N(H_0)/H_0$.

Examples:

$$\mathcal{C}(\mathbb{R}, p = e^{2\pi i\theta}, S^1) \simeq \mathbb{Z} \simeq \mathbb{Z}/\{e\} = N(\{e\})/\{e\}$$

$$\mathcal{C}(S^1, p = z^k, S^1) \simeq \mathbb{Z}_k \simeq \mathbb{Z}/k\mathbb{Z} = N(k\mathbb{Z})/k\mathbb{Z}$$

$$\mathcal{C}(\widetilde{X}, p, S^1 \vee S^1) = \text{identity} \simeq \{e\}$$



$$(\widetilde{X}, \widetilde{x_0}) \xrightarrow{h} (\widetilde{X}, h(\widetilde{x_0}))$$
$$p \searrow \quad \swarrow p$$
$$S^1 \vee S^1$$

$H_0 = p_*(\pi_1(\widetilde{X}, \widetilde{x_0})) =$
$$< a^n b a^{-n}, b^k a b^{-k} \mid n, k \in \mathbb{Z} >$$

$H = p_*(\pi_1(\widetilde{X}, \widetilde{x_1})) =$
$$< b, a^n b a^{-n}, a^{-1} b^k a b^{-k} a \mid n, k \in \mathbb{Z}, n \neq -1 >$$

Recall, $\Phi : \pi_1(X, x_0)/H_0 \to p^{-1}(x_0), \quad \Phi(H[\alpha]) = \widetilde{\alpha}(1)$

Define: $\Psi : \mathcal{C}(\widetilde{X}, p, X) \to p^{-1}(x_0), \quad \Psi(h) = h(\widetilde{x_0})$

Claim: $\Psi(\mathcal{C}(\widetilde{X}, p, X)) = \Phi(N(H_0)/H_0)$

$$\Psi(\mathcal{C}(\widetilde{X}, p, X)) = \{h(\widetilde{x_0}) \mid h \in \mathcal{C}(\widetilde{X}, p, X)\}$$

$$= \{h(\widetilde{x_0}) \mid p_*(\pi_1(\widetilde{X}, h(\widetilde{x}))) = H_0\}$$

$\widetilde{X}$ path connected implies $\exists \widetilde{\alpha}$, a path in $\widetilde{X}$ from $\widetilde{x_0}$ to $h(\widetilde{x})$

$[\alpha] \in \pi_1(X, x_0)/H_0$ and $\widetilde{\alpha}(1) = h(\widetilde{x_0})$.

I.e., α is a loop in X and $\widetilde{\alpha}$ is a path in $\widetilde{X}$ from $\widetilde{x_0}$ to $\widetilde{\alpha}(1)$

Thus by Lemma 79.3a, $\alpha\ p_*(\pi_1(\widetilde{X}, h(\widetilde{x})))\ \alpha^{-1} = H_0$.

But $h \in \mathcal{C}(\widetilde{X}, p, X)$ iff $p_*(\pi_1(\widetilde{X}, h(\widetilde{x}))) = H_0$

$H_0 = \alpha\ p_*(\pi_1(\widetilde{X}, h(\widetilde{x})))\ \alpha^{-1} = \alpha H_0 \alpha^{-1}$ iff $\alpha \in N(H_0)$.

Thus $\Psi(\mathcal{C}(\widetilde{X}, p, X)) = \Phi(N(H_0)/H_0)$

THM (81.2). The bijection $\Phi^{-1} \circ \Psi : \mathcal{C}(\widetilde{X}, p, X)) \to N(H_0)/H_0$ is an isomorphism of groups.

COR (81.3). $H_0 \lhd \pi_1(X, x_0)$ iff $\forall \widetilde{x} \in p^{-1}(x_0)$, $\exists$ covering transformation h such that $h(\widetilde{x_0}) = \widetilde{x}$.

Note that in this case $N(H_0)/H_0 = \pi_1(X, x_0)/H_0$

COR (81.4). $\widetilde{X}$ simply connected implies $\mathcal{C}(\widetilde{X}, p, X) \simeq \pi_1(X, x_0)$

The covering map $p : \widetilde{X} \to X$ is **regular** if $H_0 \lhd \pi_1(X, x_0)$

I.e., the covering map is regular if $N(H_0) = \pi_1(X, x_0)$

A (left) **group action** A of a group G on a set X is a function $A : G \times X \to X$, $A(g, x) = g(x)$ that satisfies the following two axioms:

Identity: $e(x) = x \ \ \forall x \in X$.

Compatibility: $(gh)x = g(h(x)) \ \ \forall g, h \in G$ and $\forall x \in X$.

The **orbit** of a point $x \in X = G(x) = \{g(x) \mid g \in G\}$.

The **orbit space** $X/G = X/\sim$ where $x \sim y$ iff $y \in G(x)$.

If G acts on a topological space X, the action is **properly discontinuous** if $\forall x \in X \exists$ an open neighborhood U of x in X, such that $g(U) \cap U \neq \emptyset$ implies $g = e$.

Thm (81.5). Let $\widetilde{X}$ be pc and lpc and let G be a group of homomorphisms of $\widetilde{X}$. The quotient map $\pi : \widetilde{X} \to \widetilde{X}/G$ is a covering map iff the action of G is properly discontinuous. In this case π is regular and $\mathcal{C}(\widetilde{X}, \pi, \widetilde{X}/G) \simeq G$

Thm (81.5). If $p : \widetilde{X} \to X$ is a regular covering map and $G = \mathcal{C}(\widetilde{X}, p, X)$ then $\exists$ homeomorphism $k : X/G \to X$ such that $p = k \circ \pi$

$$\begin{array}{ccc} \widetilde{X} & = & \widetilde{X} \\ \pi \downarrow & & \downarrow p \\ \widetilde{X}/G & \xrightarrow{k} & X \end{array}$$