Recall $\prod_n H^n(X;G)$ is an abelian group under addition.

$([\phi_0],[\phi_1],[\phi_2],...) \in \prod_n H^n(X;G)$ Similarly for homology

where $\phi_n : C_n \to G \;\in\; ker(\delta) \;=\; Z_n^*(X;G) \;\subset\; C^n(X;G)$

$\phi \sim \psi \in H^n(X;G)$ iff $\phi = \psi + \delta\chi$ for some $\chi \in C^{n+1}(X;G)$.

Defn: A is a **graded ring** if $A = \bigoplus_{k\geq 0} A_k$ where
A_k are additive groups with multiplication $m : A_k \times A_\ell \to A_{k+\ell}$

Defn: $a_k \in A_k$ has **grade** $=$ **degree** $=$ **dimension** $= |a_k| = k$

Examples: Polynomial rings.

$$R[x] = \{r_0 + r_1x + ... + r_kx^k \mid r_i \in R, k \in \mathbb{N}\}$$

$$R[x]/(x^n) = \{r_0 + r_1x + ... + r_kx^k \mid r_i \in R, k \in \{0,...,n-1\}\}$$

where $x^ix^j = x^m$ where $m = i + j \; mod \; n$.

$R[x_1,...,x_n]$ where $rx_1^{i_1}x_2^{i_2}...x_\ell^{i_n}$ has grade $\sum_{j=1}^n i_j$ ← Maybe

Example: Exterior Algebras $\Lambda_R[x_1,...,x_n]$

Basis for grade k: $x_{i_1}x_{i_2}...x_{i_k}$ where $i_1 < i_2 < ... < i_k$

and $x_ix_j = -x_jx_i$ and $x_i^2 = 1$ for all i,j

Maybe

$\phi : C_k \longrightarrow R$

Cup Product Let R be a commutative ring with identity.

Defn: For cochains $\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$,
the **cup product** $\phi \smile \psi \in C^{k+\ell}(X;R)$ is the cochain defined by

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|_{[v_0,...,v_k]}) \cdot \psi(\sigma|_{[v_k,...,v_{k+\ell}]}).$$

where $\sigma \in C_{k+\ell}(X)$.

The cup product is associative:

$\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, $\chi \in C^m(X;R)$ $\sigma \in C_{m+k+\ell}$

$$(\chi \smile (\phi \smile \psi))(\sigma) = \chi(\sigma|_{[v_0,...,v_m]}) \cdot (\phi \smile \psi)(\sigma|_{[v_m,...,v_{m+k+\ell}]})$$

$$= [\chi(\sigma|_{[v_0,...,v_m]}) \cdot \phi(\sigma|_{[v_m,...,v_{m+k}]})] \cdot \psi(\sigma|_{[v_{m+k},...,v_{m+k+\ell}]})$$

$$= (\chi \smile \phi)(\sigma|_{[v_0,...,v_{m+k}]}) \cdot \psi(\sigma|_{[v_{m+k},...,v_{m+k+\ell}]})$$

$$= ((\chi \smile \phi) \smile \psi)(\sigma)$$

The cup product is distributive: If $k = \ell$ $\phi, \psi \in C^k$

$$(\chi \smile (\phi + \psi))(\sigma) = \chi(\sigma|_{[v_0,...,v_m]}) \cdot (\phi + \psi)(\sigma|_{[v_m,...,v_{m+k}]})$$

$$= \chi(\sigma|_{[v_0,...,v_m]}) \cdot [\phi(\sigma|_{[v_m,...,v_{m+k}]}) + \psi(\sigma|_{[v_m,...,v_{m+k}]})]$$

$$= \chi(\sigma|_{[v_0,...,v_m]}) \cdot \phi(\sigma|_{[v_m,...,v_{m+k}]}) + \chi(\sigma|_{[v_0,...,v_m]}) \cdot \psi(\sigma|_{[v_m,...,v_{m+k}]})$$

$$= (\chi \smile \phi)(\sigma) + (\chi \smile \psi)(\sigma)$$

Similarly $((\phi + \psi) \smile \chi)(\sigma) = (\phi \smile \chi)(\sigma) + (\psi \smile \chi)(\sigma)$

Thus if R is a ring, $\bigoplus_n C^n(X;R)$ is a ring.

If R has a multiplicative identity, 1, then $\iota : C_0(X) \to R$, $\iota(v) = 1$ for all vertices v is the multiplicative identity for $\bigoplus_n C^n(X;G)$:

$$(\iota \smile \psi)(\sigma) = \iota(\sigma|_{[v_0]}) \cdot \psi(\sigma|_{[v_0,\dots,v_\ell]}) = 1 \cdot \psi(\sigma|_{[v_0,\dots,v_\ell]}) = \psi(\sigma)$$

$$(\phi \smile \iota)(\sigma) = \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \iota(\sigma|_{[v_k]}) = \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot 1 = \phi(\sigma).$$

Proposition 1. *For $\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, we have*

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi.$$

Note: The cup product of two cocycles is a cocycle. $\delta\phi = 0,\ \delta\psi = 0$

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi = 0 + 0 = 0.$$

The cup product of a cocycle and a coboundary is a coboundary.

$$(-1)^k \phi \smile \delta\psi = \overset{=0}{\delta\phi} \smile \psi + (-1)^k \phi \smile \delta\psi = \delta(\phi \smile \psi).$$

cocycle coboundary

The cup product of a coboundary and a cocycle is a coboundary.

$$\delta\phi \smile \psi = \delta\phi \smile \psi + \overset{0}{\cancel{(-1)^k}} \phi \smile \delta\psi = \delta(\phi \smile \psi).$$

coboundary cocycle

Proof: $\delta(\phi \smile \psi)$, $\delta(\phi \smile \psi)$, $\delta\phi \smile \psi \in C^{k+\ell+1}(X;R)$,

$\delta\phi \in C^{k+1}(X;R)$ and $\delta\psi \in C^{\ell+1}(X;R)$

$$(\delta\phi \smile \psi)(\sigma) = \phi(\partial(\sigma|_{[v_0,\dots,v_{k+1}]})) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$
$$= \sum_{i=0}^{k+1} (-1)^i \phi(\sigma|_{[v_0,\dots,\widehat{v}_i,\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}).$$

$(\phi \smile \delta\psi)(\sigma)$

$$= \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\partial(\sigma|_{[v_k,\dots,v_{k+\ell+1}]})) = \sum_{i=k}^{k+\ell+1} (-1)^{i-k} \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\widehat{v}_i,\dots,v_{k+\ell+1}]})$$

Thus $[\delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi](\sigma)$

$$= \sum_{i=0}^{k+1} (-1)^i \phi(\sigma|_{[v_0,\dots,\widehat{v}_i,\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$
$$+(-1)^k \sum_{i=k}^{k+\ell+1} (-1)^{i-k} \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\widehat{v}_i,\dots,v_{k+\ell+1}]})$$
$$= \sum_{i=0}^{k} (-1)^i \phi(\sigma|_{[v_0,\dots,\widehat{v}_i,\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}) + (-1)^{k+1} \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$
$$+(-1)^k \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}) + \sum_{i=k+1}^{k+\ell+1} (-1)^i \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\widehat{v}_i,\dots,v_{k+\ell+1}]})$$
$$= \sum_{i=0}^{k} (-1)^i \phi(\sigma|_{[v_0,\dots,\widehat{v}_i,\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}) + \sum_{i=k+1}^{k+\ell+1} (-1)^i \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\widehat{v}_i,\dots,v_{k+\ell}}$$
$$= \sum_{i=0}^{k+\ell+1} (-1)^i (\phi \smile \psi)(\sigma|_{[v_0,\dots,\widehat{v}_i,\dots,v_{k+\ell+1}]}) = (\phi \smile \psi)(\partial(\sigma)) = \delta(\phi \smile \psi)(\sigma)$$