

$Hom(A, G) = \{h : A \rightarrow G \mid h \text{ homomorphism}\}$
 $Hom(A, G)$ is a group under function addition.

The **dual homomorphism** to $f : A \rightarrow B$ is the homomorphism $f^* : Hom(A, G) \leftarrow Hom(B, G)$ defined by $f^*(\psi) = \psi \circ f : A \rightarrow B \rightarrow G$

That is the assignment

$$A \rightarrow Hom(A, G) \quad \text{and} \quad f \rightarrow f^*$$

is a **contravariant functor** from the category of abelian groups and homomorphisms to itself since

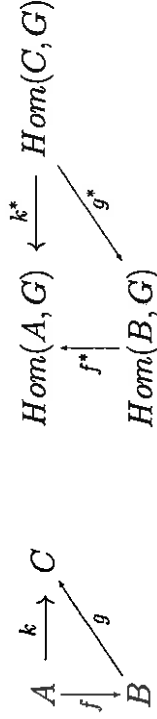
If $i : A \rightarrow A$ is the identity map on A , then

$$i_*(\psi) = \psi \circ i = \psi \text{ is the identity map on } Hom(A, G).$$

And if $f : A \rightarrow B, g : B \rightarrow C, \psi : C \rightarrow G$

$$(f^* \circ g^*)(\psi) = f^*(g^*(\psi)) = f^*(\psi \circ g) = \psi \circ g \circ f = (g \circ f)^*(\psi)$$

In other words, if the diagram on the left commutes, so does the one on the right:



- Hence f isomorphism implies f^* is an isomorphism.
- The constant fn $f = 0$ implies $f^* = 0$ since $f^*(\psi) = \psi \circ f = \psi \circ 0$.

Given a chain complex:

$$\dots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \dots$$

Its dual is also a chain complex:

$$\dots \leftarrow Hom(C_{n+1}, G) \xleftarrow{\partial_{n+1}^*} Hom(C_n, G) \xleftarrow{\partial_n^*} Hom(C_{n-1}, G) \leftarrow \dots$$

Cohomology

Cochains: $\Delta^n(X; G) = Hom(C_n, G) = \prod_{\sigma_\alpha} G$

Coboundary map: $\delta^1 = \partial_1^* : \Delta^0(X; G) \rightarrow \Delta^1(X; G)$

Cohomology: $H^n(X; G) = Z^n(X; G) / B^n(X; G) = ker(\delta_{n+1}) / im(\delta_n)$

$$n = 0: \quad Hom(\zeta_1, \zeta) \subseteq Hom(C_0, G)$$

The dual of $C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$ is $\Delta^1(X; G) \xleftarrow{\delta_1} \Delta^0(X; G) \xleftarrow{\delta_0} 0$

$im(\delta_0) = 0$. Thus $H^0(X; G) = ker(\delta_1) / im(\delta_0) = ker(\delta_1)$

$\psi : C_0 = \langle V \rangle \rightarrow G$, defined by $\psi(v_\alpha) = g_\alpha$

$$\delta_1(\psi) : C_1 = \langle E \rangle \rightarrow G,$$

$$\delta_1(\psi)([v_1, v_2]) = \psi \circ \partial([v_1, v_2]) = \psi(v_2 - v_1) = \psi(v_2) - \psi(v_1).$$

Application: ψ = elevation, $\delta_1(\psi)$ = change in elevation.

Application:

ψ = voltage at connection points, $\delta_1(\psi)$ = voltage across components.

$\delta_1(\psi) = 0$ iff

$$\delta_1(\psi)([v_1, v_2]) = \psi \circ \delta_1([v_1, v_2]) = \psi(v_2 - v_1) = \psi(v_2) - \psi(v_1) = 0$$

for all edges

Thus

$$\ker(\delta_1) = \{\psi : C_0 \rightarrow G \mid \psi \text{ is constant on the components of } X\}$$

Hence $H^0(X; G) = \prod_{\text{components of } X} G$.

Recall $H_0(X; G) = \sum_{\text{components of } X} G$.

$n = 1$:

Dual of $C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0$ is $\Delta^2(X; G) \xleftarrow{\delta_2} \Delta^1(X; G) \xleftarrow{\delta_1} \Delta^0(X; G)$

$H^1(X; G) = Z^1(X; G)/B^1(X; G) = \ker(\delta_2)/\text{im}(\delta_1)$

$\text{im}(\delta_1) = ?$

Suppose $\delta_1(\psi) = \sigma : \Delta^1 \rightarrow G$

Then σ is determined by trees in the 1-skeleton of $X = X^1$.

Let $T =$ a set of maximal trees for X^1 & let $A = \{e_a \in \Delta^1 \mid e_a \notin T\}$.

If $\Delta^2 = 0$, $H^1(X; G) = \ker(\delta_2)/\text{im}(\delta_1) = \Delta^1/\text{im}(\delta_1) = \prod_{e_a \in A} G$

Recall if $\Delta^2 = 0$, $H_1^0(X; G) = \sum_{e_a \in A} G$

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Note δ_1 is onto if X^1 is a forest = \sqcup Trees

Lemma: If $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is exact, then

$$\text{Hom}(A, G) \xleftarrow{f^*} \text{Hom}(B, G) \xleftarrow{g^*} \text{Hom}(C, G) \leftarrow 0 \text{ is exact.}$$

Proof:

Claim: g onto implies g^* is 1:1.

Suppose $g^*(\psi) = \psi \circ g = 0$. Since g is onto, $\psi(x) = 0$ for all $x \in C$. Thus $\psi = 0$ and g^* is 1:1.

Thus we have exactness at $\text{Hom}(C, G)$.

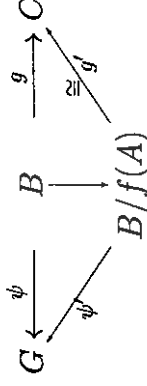
Claim: $\text{im}(f) = \ker(g)$ implies $\text{im}(g^*) = \ker(f^*)$.

$\text{im}(f) \subset \ker(g)$ implies $g \circ f = 0$

implies $f^* \circ g^* = (g \circ f)^* = 0^* = 0$ and thus $\text{im}(g^*) \subset \ker(f^*)$.

Suppose $\psi \in \ker(f^*)$, $\psi : B \rightarrow G$. Then $f^*(\psi) = \psi \circ f = 0$. Thus $\psi(f(A)) = 0$ and ψ induces homomorphism $\psi' : B/f(A) \rightarrow G$

g induces an isomorphism $g' : B/\ker(g) = B/f(A) \rightarrow C$.



$$g^*(\psi' \circ (g')^{-1}) = \psi' \circ (g')^{-1} \circ g = \psi$$

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Lemma: If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is split exact, then $0 \rightarrow \text{Hom}(A, G) \xleftarrow{f^*} \text{Hom}(B, G) \xleftarrow{g^*} \text{Hom}(C, G) \leftarrow 0$ is split exact.

Proof: $\exists \pi : B \rightarrow A$ such that $\pi \circ f = id_A$.

Thus $(\pi \circ f)^* = f^* \circ \pi^* = \text{identity on } \text{Hom}(A, G)$.

Thus f^* is surjective and the dual sequence splits.

Note also that $\text{Hom}(\bigoplus A_\alpha, G) = \prod \text{Hom}(A_\alpha, G)$,

and thus $\text{Hom}(A \oplus C, G) = \text{Hom}(A, G) \oplus \text{Hom}(C, G)$

Example: The dual of the exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{x^2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \rightarrow 0$

$$0 \leftarrow \text{Hom}(\mathbb{Z}, G) \xleftarrow{f^*} \text{Hom}(\mathbb{Z}, G) \xleftarrow{\pi^*} \text{Hom}(\mathbb{Z}_2, G) \leftarrow 0$$

$\pi^*(\psi) = \psi \circ \pi : \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \xrightarrow{\psi} G$, defined by $(\psi \circ \pi)(1) = \psi(1)$.

$t^*(\psi) = \psi \circ t : \mathbb{Z} \xrightarrow{t} \mathbb{Z} \xrightarrow{\psi} G$,

defined by $(\psi \circ t)(1) = \psi(2) = \psi(1) + \psi(1) = 2\psi(1)$.

not onto if $G = \mathbb{Z}$

Defn: A free resolution of an abelian group H is an exact sequence of abelian groups,

$$\dots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$$

where each F_i is free.

$$F_i = \langle v_\alpha \mid \alpha \in A \rangle$$

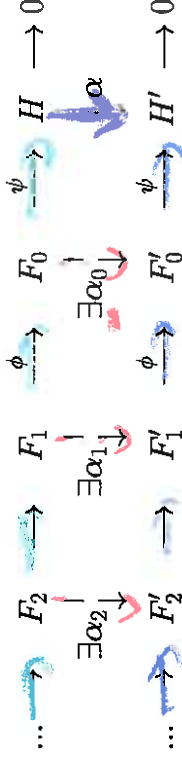
Recall an exact sequence is a chain complex, and the dual of a chain complex is a chain complex.

Thus the dualization of this free resolution is a chain complex:

$$\dots \xleftarrow{f_2^*} \text{Hom}(F_1, G) \xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \leftarrow 0$$

Let $H^n(F; G) = \text{Ker}(f_{n+1}^*) / \text{im}(f_n^*)$ *temp notation*

Lemma 3.1: a.) Given two free resolutions F and F' of H and H' , respectively, every homomorphism $\alpha : H \rightarrow H'$ can be extended to a chain map from F to F' .



Furthermore, any two such chain maps extending α are chain homotopic.

b.) For any two free resolutions F and F' of H , \exists canonical isomorphism $H^n(F; G) = H^n(F'; G)$ for all n .

Example: A short exact sequence of abelian groups,

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow F_1 \rightarrow F_0 \rightarrow H \rightarrow 0$$

where F_i are free is called a **free resolution of H** .

Example: $0 \rightarrow B_p(X) \hookrightarrow Z_p(X) \rightarrow H_p(X) \rightarrow 0$

Example:

Let F_0 be the free abelian group generated by the generators of H .

Let F_1 be kernel of projection map $F_0 \rightarrow H$.

Dual of the exact seq $0 \rightarrow F_1 \rightarrow F_0 \rightarrow H \rightarrow 0$ is the chain complex:

$$0 \xleftarrow{f_2^*} \text{Hom}(F_1, G) \xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \xleftarrow{0} 0$$

Recall $F_1 \rightarrow F_0 \rightarrow H \rightarrow 0$ exact implies its dual is also exact:

$$\text{Hom}(F_1, G) \xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \xleftarrow{0} 0$$

Note $H^n(F; G) = \text{Ker}(f_{n+1}^*) / \text{im}(f_n^*) = 0$ for $n > 1$.

And $H^0(F; G) = \text{Ker}(f_1^*) / \text{im}(f_0^*) = 0$.

But $H^1(F; G) = \text{Ker}(f_2^*) / \text{im}(f_1^*) = \text{Hom}(F_1, G) / \text{im } f_1^*$

Definition: $\text{Ext}(H, G) = H^1(F; G)$ (the extension of G by H).

$$\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$$

For computational purposes, the following properties are useful.

(a) $\text{Ext}(H \oplus H', G) \cong \text{Ext}(H, G) \oplus \text{Ext}(H', G)$ since the direct sum of free resolutions is the free resolution of the direct sum.

(b) $\text{Ext}(H, G) \cong 0$ if H is free since $0 \rightarrow H \rightarrow H \rightarrow 0$ is a free resolution of H .

(c) $\text{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$

by dualizing the free resolution $0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0$.

to produce the exact sequence:

$$0 \leftarrow \text{Ext}(\mathbb{Z}_n, G) \xleftarrow{\cong} \text{Hom}(\mathbb{Z}, G) \xleftarrow{n} \text{Hom}(\mathbb{Z}, G) \xleftarrow{\cong} \text{Hom}(\mathbb{Z}_n, G) \xleftarrow{0} 0$$

Take free G

Theorem 1. If a chain complex C_\bullet of free abelian groups has homology groups $H_\bullet(C)$, then the cohomology groups $H^n(C; G)$ of the cochain complex $\text{Hom}(C_\bullet, G)$ are determined by the split exact sequences

$$0 \rightarrow \text{Ext}(H_{n-1}(C), G) \rightarrow H^n(C; G) \xrightarrow{h} \text{Hom}(H_n(C), G) \rightarrow 0$$

Corollary 1. If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_{n-1}$, then

$$H^n(C; \mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}$$

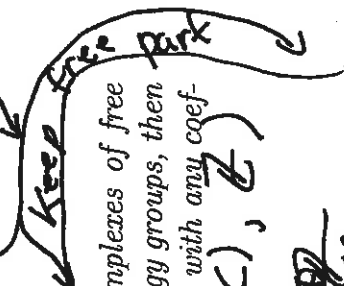
Corollary 2. If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G .

$$\text{Hom}(H_n(C), \mathbb{Z})$$

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$$\text{Hom}(\bigoplus_n \mathbb{Z} \oplus (\bigoplus \mathbb{Z}_n), \mathbb{Z}) \cong \bigoplus_n \mathbb{Z} \oplus T$$

$$F \oplus T$$



$$\mathbb{Z} \xrightarrow{m} \mathbb{Z}$$

$$G \xrightarrow{n} B \quad n^*(\varphi)(1) = \varphi \circ n(1) = \varphi(n)$$

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where F_i are free is called a **free resolution of H** .

Example: $0 \rightarrow B_p(X) \hookrightarrow Z_p(X) \rightarrow H_p(X) \rightarrow 0$

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Recall $F_1 \rightarrow F_0 \rightarrow H \rightarrow 0$ exact implies its dual is also exact:

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Note $H^n(F; G) = \text{Ker}(f_{n+1}^*)/\text{im}(f_n^*) = 0$ for $n > 1$.

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But $H^1(F; G) = \text{Ker}(f_2^*)/\text{im}(f_1^*) = ?$. $\text{Hom}(F_1, G)/\text{im } f_1^*$

Definition: $\text{Ext}(H, G) = H^1(F, G)$ (the extension of G by H).

$$\varphi: \mathbb{Z} \rightarrow G \quad f_2^*(\varphi) = \varphi \circ f_2: 0 \rightarrow G$$

$$\text{Hom}(0, G) \xleftarrow{f_2^*} \text{Hom}(\mathbb{Z}, G)$$

$$\textcircled{1} \quad 0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0$$

$$\textcircled{2} \quad \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \text{Hom}(C)) \xleftarrow{m^*} \text{Hom}(C) \xleftarrow{f_0^*} \text{Hom}(C) \leftarrow 0$$

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by dualizing the free resolution $0 \rightarrow \mathbb{Z} \xrightarrow{f_1} \mathbb{Z} \xrightarrow{f_0} \mathbb{Z}_n \rightarrow 0$.

to produce the exact sequence: $f_1^* \quad f_0^*$

$$0 \leftarrow \text{Ext}(\mathbb{Z}_n, G) \xleftarrow{f_1^*} \text{Hom}(\mathbb{Z}, G) \xleftarrow{f_0^*} \text{Hom}(\mathbb{Z}, G) \leftarrow \text{Hom}(\mathbb{Z}_n, G) \leftarrow 0$$

$$0 \leftarrow G/nG \leftarrow G \leftarrow G \leftarrow 0$$

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$$= \text{Ext}(H_0, G) = \text{Hom}(\mathbb{Z}_n, G) / \text{im } f_1^*$$