$Hom(A,G) = \{h : A \to G \mid h \text{ homomorphism } \}$

$Hom(A,G)$ is a group under function addition.

The **dual homomorphism to** $f : A \to B$ is the
homomorphism $f^* : Hom(A,G) \leftarrow Hom(B,G)$
defined by $f^*(\psi) = \psi \circ f : A \to B \to G$

That is the assignment

$$A \to Hom(A,G) \quad \text{and} \quad f \to f^*$$

is a **contravarient functor** from the category of abelian groups and homomorphisms to itself since

If $i : A \to A$ is the identity map on A, then
$i_*(\psi) = \psi \circ i = \psi$ is the identity map on $Hom(A,G)$.

And if $f : A \to B, \quad g : B \to C, \quad \psi : C \to G$

$$(f^* \circ g^*)(\psi) = f^*(g^*(\psi)) = f^*(\psi \circ g) = \psi \circ g \circ f = (g \circ f)^*(\psi)$$

In other words, if the diagram on the left commutes, so does the one on the right:

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "f"'] & C \\ B \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} Hom(A,G) & Hom(C,G) \arrow[l, "k^*"'] \arrow[dl, "g^*"] \\ Hom(B,G) \arrow[u, "f^*"] & \end{tikzcd}$$

- Hence f isomorphism implies f^* is an isomorphism.

- The constant fn $f = 0$ implies $f^* = 0$ since $f^*(\psi) = \psi \circ f = \psi \circ 0$.

Given a chain complex:

$$\ldots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \ldots$$

Its dual is also a chain complex:

$$\ldots \leftarrow Hom(C_{n+1},G) \xleftarrow{\partial^*_{n+1}} Hom(C_n,G) \xleftarrow{\partial^*_n} Hom(C_{n-1},G) \leftarrow \ldots$$

Cohomology

Cochains: $\Delta^n(X;G) = Hom(C_n,G) = \prod_{\sigma_\alpha} G$

Coboundary map: $\delta^1 = \partial_1^* : \Delta^0(X;G) \to \Delta^1(X;G)$

Cohomology: $H^n(X;G) = Z^n(X;G)/B^n(X;G) = ker(\delta_{n+1})/im(\delta_n)$

<u>$n = 0$</u>: $\quad Hom(C_1; G) \leftarrow Hom(C_0, G)$

The dual of $C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$ is $\Delta^1(X;G) \xleftarrow{\delta_1} \Delta^0(X;G) \xleftarrow{\delta_0} 0$

$im(\delta_0) = 0$. Thus $H^0(X;G) = ker(\delta_1)/im(\delta_0) = ker(\delta_1)$

$\psi : C_0 = < V > \longrightarrow G$, defined by $\psi(v_\alpha) = g_\alpha$

$\delta_1(\psi) : C_1 = < E > \longrightarrow G$,

$\delta_1(\psi)([v_1,v_2]) = \psi \circ \partial([v_1,v_2]) = \psi(v_2 - v_1) = \psi(v_2) - \psi(v_1)$.

Application: ψ = elevation, $\delta_1(\psi)$ = change in elevation.
Application:
ψ = voltage at connection points, $\delta_1(\psi)$ = voltage across components.

$\delta_1(\psi) = 0$ iff

$\delta_1(\psi)([v_1, v_2]) = \psi \circ \delta_1([v_1, v_2]) = \psi(v_2 - v_1) = \psi(v_2) - \psi(v_1) = 0.$
for all edges

Thus

$ker(\delta_1) = \{\psi : C_0 \to G \mid \psi \text{ is constant on the components of } X\}$

Hence $H^0(X; G) = \prod_{\text{components of } X} G.$

Recall $H_0(X; G) = \sum_{\text{components of } X} G.$

$\underline{n = 1}$:

Dual of $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ is $\Delta^2(X; G) \xleftarrow{\delta_2} \Delta^1(X; G) \xleftarrow{\delta_1} \Delta^0(X; G)$

$H^1(X; G) = Z^1(X; G)/B^1(X; G) = ker(\delta_2)/im(\delta_1)$

$im(\delta_1) = ?$

Suppose $\delta_1(\psi) = \sigma : \Delta^1 \to G$

$\Downarrow$

Then σ is determined by trees in the 1-skeleton of $X = X^1$.

Let T = a set of maximal trees for X^1 & let $A = \{e_a \in \Delta^1 \mid e_a \notin T\}$.

If $\Delta^2 = 0$, $H^1(X; G) = ker(\delta_2)/im(\delta_1) = \Delta^1/im(\delta_1) = \prod_{e_\alpha \in A} G$

Recall if $\Delta^2 = 0$, $H_1(X; G) = \sum_{e_\alpha \in A} G$

Lemma: If $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, then
$Hom(A, G) \xleftarrow{f^*} Hom(B, G) \xleftarrow{g^*} Hom(C, G) \leftarrow 0$ is exact.

Proof:

Claim: g onto implies g^* is 1:1.

Suppose $g^*(\psi) = \psi \circ g = 0$. Since g is onto, $\psi(x) = 0$ for all $x \in C$. Thus $\psi = 0$ and g^* is 1:1.

Thus we have exactness at $Hom(C, G)$.

Claim: $im(f) = ker(g)$ implies $im(g^*) = ker(f^*)$.

$im(f) \subset ker(g)$ implies $g \circ f = 0$
implies $f^* \circ g^* = (g \circ f)^* = 0^* = 0$ and thus $im(g^*) \subset ker(f^*)$.

Suppose $\psi \in ker(f^*)$, $\psi : B \to G$. Then $f^*(\psi) = \psi \circ f = 0$. Thus $\psi(f(A)) = 0$ and ψ induces homomorphism $\psi' : B/f(A) \to G$

g induces an isomorphism $g' : B/ker(g) = B/f(A) \to C$.

$$\begin{tikzcd}
G & B \arrow[l, "\psi"'] \arrow[r, "g"] \arrow[d] & C \\
 & B/f(A) \arrow[ul, "\psi'"] \arrow[ur, "\cong", "g'"'] &
\end{tikzcd}$$

$g^*(\psi' \circ (g')^{-1}) = \psi' \circ (g')^{-1} \circ g = \psi$

Note δ_1 is onto if X^1 is a forest $= \sqcup$ Trees

π

Lemma: If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is split exact, then $0 \leftarrow Hom(A,G) \xleftarrow{f^*} Hom(B,G) \xleftarrow{g^*} Hom(C,G) \leftarrow 0$ is split exact.

Proof: $\exists \pi : B \to A$ such that $\pi \circ f = id_A$.

Thus $(\pi \circ f)^* = f^* \circ \pi^* =$ identity on $Hom(A,G)$.

Thus f^* is surjective and the dual sequence splits.

Note also that $Hom(\bigoplus A_\alpha, G) = \prod Hom(A_\alpha, G)$,

and thus $Hom(A \bigoplus C, G) = Hom(A,G) \bigoplus Hom(C,G)$

not onto

not exact

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t

Example: The dual of the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \to 0$

$$0 \leftarrow Hom(\mathbb{Z}, G) \xleftarrow{t^*} Hom(\mathbb{Z}, G) \xleftarrow{\pi^*} Hom(\mathbb{Z}_2, G) \leftarrow 0$$

$\pi^*(\psi) = \psi \circ \pi : \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \xrightarrow{\psi} G$, defined by $(\psi \circ \pi)(1) = \psi(1)$.

$\times 2$

$t^*(\psi) = \psi \circ t : \mathbb{Z} \xrightarrow{t} \mathbb{Z} \xrightarrow{\psi} G$,

defined by $(\psi \circ t)(1) = \psi(2) = \psi(1) + \psi(1) = 2\psi(1)$.

not onto if $G = \mathbb{Z}$

Defn: A **free resolution** of an abelian group H is an exact sequence of abelian groups,

$$\ldots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where each F_i is free. $F_i = \langle v_\alpha \mid \alpha \in A \rangle$

Recall an exact sequence is a chain complex, and the dual of a chain complex is a chain complex.

Thus the dualization of this free resolution is a chain complex:

$$\ldots \xleftarrow{f_2^*} Hom(F_1, G) \xleftarrow{f_1^*} Hom(F_0, G) \xleftarrow{f_0^*} Hom(H, G) \leftarrow 0$$

Let $H^n(F; G) = Ker(f_{n+1}^*)/im(f_n^*)$ temp notation

Lemma 3.1: a.) Given two free resolutions F and F' of H and H', respectively, every homomorphism $\alpha : H \to H'$ can be extended to a chain map from F to F':

$$\begin{array}{ccccccccc}
\ldots \to & F_2 & \to & F_1 & \xrightarrow{\phi} & F_0 & \xrightarrow{\psi} & H & \to 0 \\
 & \downarrow \exists \alpha_2 & & \downarrow \exists \alpha_1 & & \downarrow \exists \alpha_0 & & \downarrow \alpha & \\
\ldots \to & F_2' & \to & F_1' & \xrightarrow{\phi} & F_0' & \xrightarrow{\psi} & H' & \to 0
\end{array}$$

Furthermore, any two such chain maps extending α are chain homotopic.

b.) For any two free resolutions F and F' of H, $\exists$ canonical isomorphism $\boxed{H^n(F; G) = H^n(F', G)}$ for all n.

Example: A short exact sequence of abelian groups,

$$\cdots \to 0 \to 0 \to 0 \to F_1 \to F_0 \to H \to 0$$

where F_i are free is called a **free resolution of** H.

Example: $0 \to B_p(X) \hookrightarrow Z_p(X) \to H_p(X) \to 0$

Example:

Let F_0 = the free abelian group generated by the generators of H.

Let F_1 = kernel of projection map $F_0 \to H$.

Dual of the exact seq $0 \to F_1 \to F_0 \to H \to 0$ is the chain complex:

$$0 \leftarrow 0 \xleftarrow{f_2^*} Hom(F_1, G) \xleftarrow{f_1^*} Hom(F_0, G) \xleftarrow{f_0^*} Hom(H, G) \leftarrow 0$$

Recall $F_1 \to F_0 \to H \to 0$ exact implies its dual is also exact:

$$Hom(F_1, G) \xleftarrow{f_1^*} Hom(F_0, G) \xleftarrow{f_0^*} Hom(H, G) \leftarrow 0$$

Note $H^n(F; G) = Ker(f_{n+1}^*)/im(f_n^*) = 0$ for $n > 1$.

And $H^0(F; G) = Ker(f_1^*)/im(f_0^*) = 0$.

But $H^1(F; G) = Ker(f_2^*)/im(f_1^*) = Hom(F_1, G)/im\, f_1^*$

Definition: $Ext(H, G) = H^1(F; G)$ (the extension of G by H).

$\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$

For computational purposes, the following properties are useful.

(a) $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$ since the direct sum of free resolutions is the free resolution of the direct sum.

(b) $\mathrm{Ext}(H, G) = 0$ if H is free $\boxed{0 \text{ since } 0 \to H \to H \to 0}$ is a free resolution of H.

(c) $\mathrm{Ext}(\mathbb{Z}/nG) \cong G/nG$ by dualizing the free resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$.

to produce the exact sequence: Take free G

$$0 \leftarrow Ext(\mathbb{Z}_n, G) \leftarrow Hom(\mathbb{Z}, G) \xleftarrow{n} Hom(\mathbb{Z}, G) \leftarrow Hom(\mathbb{Z}_n, G) \leftarrow 0$$

$$0 \leftarrow G/nG \leftarrow G \xleftarrow{n} G \leftarrow 0 \leftarrow 0$$

Theorem 1. *If a chain complex $C_\bullet$ of free abelian groups has homology groups $H_\bullet(C)$, then the cohomology groups $H^\bullet(C; G)$ of the cochain complex* $\mathrm{Hom}(C_\bullet, G)$ *are determined by the split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \xrightarrow{h} \boxed{\mathrm{Hom}(H_n(C), G)} \longrightarrow 0.$$

Corollary 1. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_n$, then*

$$H^n(C; Z) \cong \boxed{(H_n/T_n)} \oplus T_{n-1}$$

Keep free part

Corollary 2. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G.*

$Hom(H_n(C), \mathbb{Z})$

$\cancel{Hom(\oplus \mathbb{Z} \oplus \mathbb{Z}_{p_i}}$

$Hom(\bigoplus_n \mathbb{Z} \oplus (\oplus \mathbb{Z}_{p_i}), \mathbb{Z}) \cong G^n$

$F \oplus T$

$\mathbb{Z} \xrightarrow{n} \mathbb{Z}$

$G \xleftarrow{n^*} G$ $\qquad n^*(\varphi)(1) = \varphi \circ n(1) = \varphi(n) = n\varphi(1)$

$\varphi \to n\varphi$

Example: A short exact sequence of abelian groups,

$$0 \to F_1 \to F_0 \to H \to 0$$

where F_i are free is called a **free resolution of** H.

Example: $0 \to B_p(X) \hookrightarrow Z_p(X) \to H_p(X) \to 0$

Example:

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And $H^0(F; G) = Ker(f_1^*)/im(f_0^*) = 0$.

But $H^1(F; G) = Ker(f_2^*)/im(f_1^*) =$?. $Hom(F_1, G)/im f_1^*$

Definition: $Ext(H, G) = H^1(F; G)$ $\qquad$ (the extension of G by H).

$f_2^*(\varphi) = \varphi \circ f_2 : 0 \to G$

$\varphi : \mathbb{Z} \to G$

$Hom(0, G) \xleftarrow{f_2^*} Hom(\mathbb{Z}, G)$

(1) $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$

$0 \leftarrow Hom(\,)/im(n^*) \leftarrow Hom(\,) \xleftarrow{n^*} Hom(\,) \leftarrow Hom(\,) \leftarrow 0$

For computational purposes, the following properties are useful.

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(b) $\text{Ext}(H, G) = 0$ if H is free
since $0 \to H \to H \to 0$ is a free resolution of H.

(c) $\text{Ext}(\mathbb{Z}/n, G) \cong G/nG$
by dualizing the free resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$.
$f_2 \quad f_1 \quad f_0$

to produce the exact sequence: $f_1^* \qquad f_0^*$

$$0 \leftarrow Ext(\mathbb{Z}_n, G) \xleftarrow{f_2^*} Hom(\mathbb{Z}, G) \xleftarrow{n} Hom(\mathbb{Z}, G) \leftarrow Hom(\mathbb{Z}_n, G) \leftarrow 0$$

$\cong \qquad \cong \qquad \cong$

$0 \leftarrow G/nG \leftarrow G \leftarrow G \leftarrow 0$

Theorem 1. *If a chain complex $C_\bullet$ of free abelian groups has homology groups $H_\bullet(C)$, then the cohomology groups $H^\bullet(C; G)$ of the cochain complex* $\text{Hom}(C_\bullet, G)$ *are determined by the split exact sequences*

$$0 \longrightarrow \text{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \xrightarrow{h} \text{Hom}(H_n(C), G) \longrightarrow 0.$$

Corollary 1. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_n$, then*

$$H^n(C; Z) \cong (H_n/T_n) \oplus T_{n-1}.$$

Corollary 2. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G.*

$Ext(\mathbb{Z}_n, G) = H^1(F, G) = Ker f_2^* / im f_1^*$

$= Hom(\mathbb{Z}, G)/im f_1^*$