singular

Proposition 1. *Let X be a CW complex.*

1. $H_k(X^n, X^{n-1}) \cong \begin{cases} 0 & n \neq k \\ \mathbb{Z}^\ell & n = k \end{cases}$

   where ℓ is the number of n-cells of X (potentially infinite).

2. *$H_k(X^n) = 0$ if $k > n$. In particular, if X is finite dimensional, then $H_k(X) = 0$ for $k > \dim X$.*

3. *The inclusion $i : X^n \hookrightarrow X$ induces isomorphisms $i_* : H_k(X^n) \to H_k(X)$ for $k < n$.*

induction

Proof. a.) $H_k(X^n, X^{n-1}) \cong H_k(X^n/X^{n-1}) \cong \vee S^n$

# of n-cells = # of copies of S^n

b.) $H_{k+1}(X^{n+1}, X^n) \dashrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow H_k(X^n, X^{n-1})$

and $H_k(X^0) = 0$ for $k > 0$ and for $k \neq n$, $H_k(X^{n-1}) \cong H_k(X^n)$. □

generators of $H_k(X^n, X^{n-1})$ = n-cells

Given a CW complex X, $H_n^{CW}(X)$, *cellular homology*, is the homology of the chain complex

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots$$

Note that $H_n(X^n, X^{n-1}) = C_n^{CW}(X)$

$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$ where

$d_{\alpha\beta}$ is the degree of the map $S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$

Theorem 1. $H^{CW}_\bullet(X) \cong H_\bullet(X)$.

The following computations follow.

$$H_\bullet(\Sigma_g) \cong \mathbb{Z}_{(0)} \oplus \mathbb{Z}^{2g}_{(1)} \oplus \mathbb{Z}_{(2)}$$
$$H_\bullet(N_g) \cong \mathbb{Z}_{(0)} \oplus (\mathbb{Z}^{g-1} \oplus \mathbb{Z}/2)_{(1)}$$
$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ and if } k = n \text{ is odd,} \\ \mathbb{Z}/2 & \text{if } k \text{ is odd and } 0 < k < n, \\ 0 & \text{otherwise.} \end{cases}$$
$$H_k(L_m(\ell_1, \ldots, \ell_n)) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } 2n-1, \\ \mathbb{Z}/m & \text{if } k \text{ is odd and } 0 < k < 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Defn: a *Moore space*, denoted by $M(G, n)$, is a simply connected CW complex X satisfying $H_n(X) = G$ and $\widetilde{H}_i(X) \cong 0$ for $i \neq n$.

Example: If G is finitely generated, let $X = (\vee S^n) \cup (\sqcup e^{n+1}_\alpha)$.

Theorem 2. *For finite CW complexes X, the Euler characteristic is*

$$\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X^n, X^{n-1}) = \sum_n (-1)^n \operatorname{rank} H_n(X).$$

For example,

$$\chi(\Sigma_g) = 2 - 2g, \qquad \chi(N_g) = 2 - g.$$