$C_n(X)/C_n(A)$

Obvious Lemma: $0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{q} C_n(X,A) \to 0$ is a short exact sequence.

Thus this short exact sequence induces a long exact sequence:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \longrightarrow \cdots$$

Example: If $A = \{p\}$, then

$$\cdots \longrightarrow H_n(\{p\}) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,\{p\}) \xrightarrow{\partial_*} H_{n-1}(\{p\}) \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,\{p\}) \xrightarrow{\partial_*} 0 \longrightarrow \cdots$$

Thus $H_n(X,\{p\}) = H_n(X)$ for $n > 0$.

By defn of H_0, $H_0(X,\{p\}) = \widetilde{H_0}(X)$.

Thus $H_n(X,\{p\}) = \widetilde{H_n}(X)$ for all n.

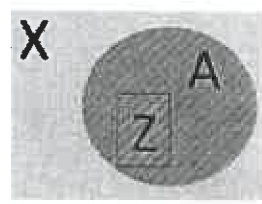
If $A \subset B \subset X$, then $0 \to C_n(B,A) \xrightarrow{i} C_n(X,A) \xrightarrow{q} C_n(X,B) \to 0$ is a short exact sequence.

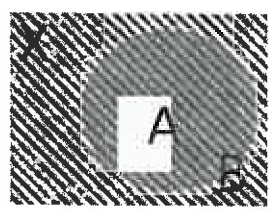
Thus this short exact sequence induces a long exact sequence of the triple (X,B,A):

$$\cdots \longrightarrow H_n(B,A) \xrightarrow{i_*} H_n(X,A) \xrightarrow{q_*} H_n(X,B) \xrightarrow{\partial_*} H_{n-1}(B,A) \longrightarrow \cdots$$

Note if A is a deformation retract of B, then $H_k(B,A) = 0$ for all k and $H_n(X,A) \cong H_n(X,B)$.

$C_n(B)/C_n(A)$





Thm 2.20 (Excision Theorem).

1.) If $\overline{Z} \subset A^o \subset X$, then $i : (X - Z, A - Z) \hookrightarrow (X,A)$

induces isomorphism: $H_n(X - Z, A - Z) \cong H_n(X,A)$

2.) If $X = A^o \cup B^o$, then $i : (B, A \cap B) \hookrightarrow (X,A)$

induces isomorphism: $H_n(B, A \cap B) \cong H_n(X,A)$

Note (1) and (2) are equivalent (let $B = X - Z$).

Defn: The pair (X,A) is **good** if $A \neq \emptyset$ and $\exists V$ such that $\overline{A} \subset V^o \subset X$ and V deformation retracts onto A.

Thm: If (X,A) is good, then $H_n(X,A) \cong \widetilde{H_n}(X/A)$.

Proof:

$$\begin{array}{ccccc}
H_n(X,A) & \xrightarrow[\text{LES of triple}]{\cong} & H_n(X,V) & \xleftarrow[\text{excision}]{\cong} & H_n(X-A, V-A) \\
{\scriptstyle q_*}\downarrow \cong & & {\scriptstyle q_*}\downarrow \cong & & {\scriptstyle q_*}\downarrow \cong \\
H_n(X/A, A/A) & \xrightarrow[\text{LES of triple}]{\cong} & H_n(X/A, V/A) & \xleftarrow[\text{excision}]{\cong} & H_n(X/A - A/A, V/A - A/A)
\end{array}$$

$\widetilde{H}_n(X/A) \cong$