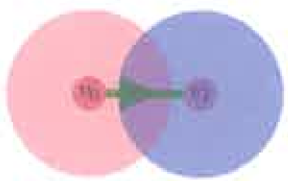
What does a generator of C_n embedded in a topological space X look like?

Let D^n be either
the closed n–dimensional disk $= \{x \in R^n \mid x \leq 1\}$ (for CW complex) or
the closed n–simplex = convex hull of $n+1$ affinely independent points.

For a Cech complex: $\bigcap_{i=0}^{n} U_i$ is visualized via simplicial R^n.

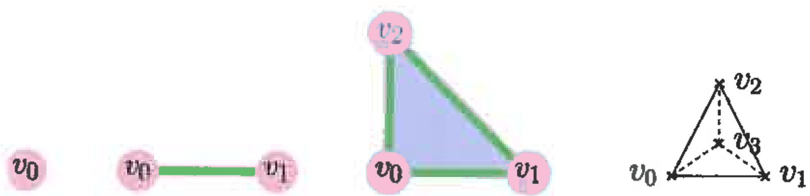


An abstract n-simplex:

$(v_0, ..., v_n)$ is visualized via simplicial R^n.

For a simplicial n-simplex:

$\Phi_\alpha : D^n_\alpha \to X$ is a homeomorphism.



For a Δ-simplex:

Assuming $\sigma_\beta : D^{n-1}_\beta \to X$ defined for all $n-1$-simplices,
$\sigma_\alpha : D^n_\alpha \to X$ is an extension of $\sigma_{\beta_i} : D^{n-1}_{\beta_i} \to X$
for all faces $D^{n-1}_{\beta_i}$ of D^n_α.
and $\sigma_\alpha : \mathring{D}^n_\alpha \to \sigma_\alpha(\mathring{D}^n_\alpha) \subset X$ is a homeomorphism.

Note that the restriction of the attaching maps
$\sigma_{\beta_i} : \mathring{D}^n_{\beta_i} \to \sigma_{\beta_i}(\mathring{D}^n_{\beta_i})$ are homeomorphisms.

$$\sqcup D^n \hookrightarrow X^{n-1} \cup \bigsqcup_\alpha D^n_\alpha \to X^{n-1} \cup \bigsqcup_\alpha D^n_\alpha / \sim \; = X^n \hookrightarrow X$$

For CW simplex:

The attaching maps $\phi_\alpha : \partial D^n_\alpha \to X^{n-1}$ are continuous.

The characteristic map $\Phi_\alpha : D^n_\alpha \to X$ extends
the attaching map $\phi_\alpha : \partial D^n_\alpha \to X^{n-1}$

and $\Phi_\alpha : \mathring{D}^n_\alpha \to \Phi_\alpha(\mathring{D}^n_\alpha) \subset X$ is a homeomorphism.

$$\sqcup D^n \hookrightarrow X^{n-1} \cup \bigsqcup_\alpha D^n_\alpha \to X^{n-1} \cup \bigsqcup_\alpha D^n_\alpha / \sim \; = X^n \hookrightarrow X$$

where $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$

For a singular simplex: $\sigma_\alpha : D^n_\alpha \to X$ is a continuous map.

$\{f_p : \{p\} \to X \mid p \in X\} \subseteq$ generators of C_0 for singular homology

What does a generator of C_0 embedded in a topological space X look like?

- simplicial/Δ/CW/singular 1-simplex: A point.

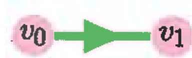
What does a generator (including boundary) of C_1 embedded in a topological space X look like?

- Simplicial 1-simplex:

- Δ 1-simplex:

  Case 1: 1 vertex

  Case 2: 2 vertices

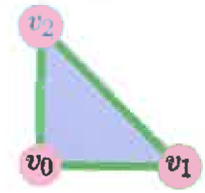


- CW 1-simplex: The same as a Δ complex.
- Singular 1-simplex: $f : [0, 1] \to X$, where f is continuous.

  I.e., a singular 1-simplex is a path in X. Note this includes the constant path.

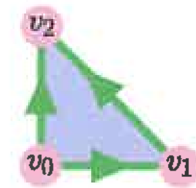
What does a generator (including boundary) of C_2 embedded in a topological space X look like?

- Simplicial 2-simplex:



- Δ 2-simplex:

  Each edge of the 2-simplex must be glued via an orientation preserving homeomorphism to a loop or edge of X^{n-1}.



  Case 1: 1 vertex

  Case 1a: 1 edge (Dunce hat)

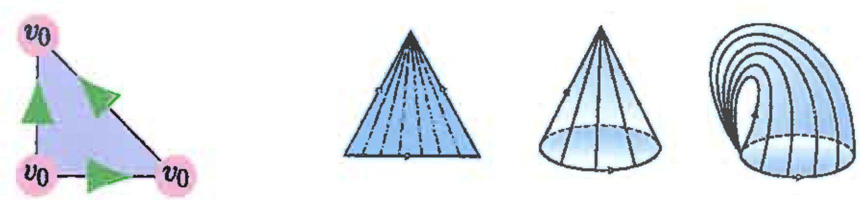
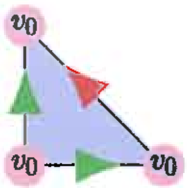


Figure 1: Identify the 2 circles. Dunce hat from math.stackexchange.com/questions/244885/dunce-hat-is-simply-connected by Ronnie Brown

See also Topological Dunce Hat by Jos Luis Rodrguez Blancas: https://youtu.be/34j4CppfRTA

Case 1bi: 2 edges



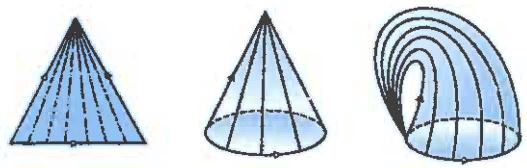
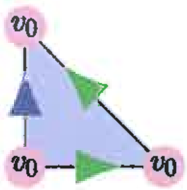
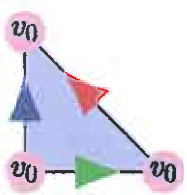


Figure 2: Don't identify the 2 circles math.stackexchange.com/questions/244885/dunce-hat-is-simply-connected by Ronnie Brown

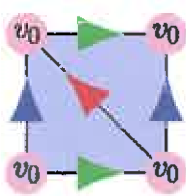
Case 1bii: 2 edges



Case 1c: 3 edges



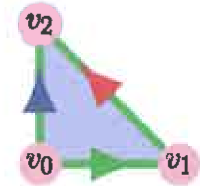
Example: torus



Case 2: 2 vertices

Case 3: 3 vertices.

Note each pair of distinct vertices defines a unique edge.



Example: Torus

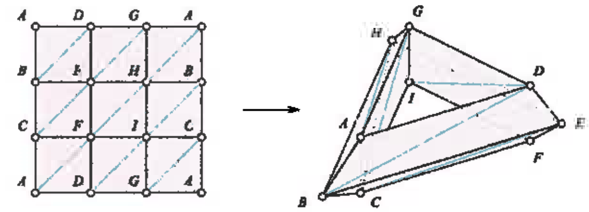


Figure III.2: A vertex map and its induced simplicial map from the square to the torus.

Figure 3: The simplicial triangulation of the torus is also a Δ-complex. From: Exterior, tex.stackexchange.com/questions/217645/typesetting-triangulations

- CW 2-simplex:

  Multiple possibilities.

  Only need attaching maps $\phi_\alpha : \partial D^2_\alpha \to X^1$ to be continuous where $D^2 = \{x \in R^2 \mid ||x|| \leq 1\}$

  For example,

- Singular 2-simplex: $f : \Delta^2 \to X$, where f is continuous.

  I.e., the image a singular 2-simplex is the image of a triangle in X. Note this image is a point if f is a constant map.

no glueing
no attaching maps
just looking at all cont fns $\Delta^n \to X$

Singular homology

Generators of $C_n(X) = \{\sigma_\alpha : \Delta^n_\alpha \to X \mid \sigma_\alpha \text{ is continuous}\}$.

Let $\sigma : (v_0, ..., v_n) \to X$ be continuous.

$$\partial_n(\sigma) = \sum_{i=1}^{n}(-1)^i \sigma|(v_0, ..., \widehat{v_i}, ..., v_n) \in C_{n-1}(X)$$

Thus $\partial^2 = 0$ and $H_n(X) = Z_n(X)/B_n(X)$ is well defined where

$Z_n = ker(\partial_n) =$ cycles and $B_n = im(\partial_{n+1}) =$ boundaries.

Suppose $f : X \to Y$ is continuous.

f induces the homomorphism $f_\# : C_n(X) \to C_n(Y)$

$f_\#(\sigma : \Delta \to X) = f \circ \sigma : \Delta \to Y$ and extend linearly.

Note: $f_\# \circ \partial = \partial \circ f_\#$

$$\begin{aligned} f_\#(\partial_n(\sigma)) &= f_\#(\sum_{i=1}^{n}(-1)^i \sigma|(v_0, ..., \widehat{v_i}, ..., v_n)) \\ &= \sum_{i=1}^{n}(-1)^i (f_\#\sigma|(v_0, ..., \widehat{v_i}, ..., v_n)) \\ &= \partial_n(f_\#(\sigma)) \end{aligned}$$

If σ is a cycle, then $f_\#(\sigma)$ is a cycle. Thus $f_\#(Z_n(X)) \subset Z_n(Y)$.

If $\sigma = \partial(\beta)$, then $f_\#(\sigma) = f_\#(\partial(\beta)) = \partial(f_\#(\beta))$

Thus $f_\#(B_n(X)) \subset B_n(Y)$.

$f_\#: \frac{Z_n(X)}{B_n(X)} \to \frac{Z_n(Y)}{B_n(X)}$

Hence $f_\# : C_n(X) \to C_n(Y)$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$

If $f : X \to Y$ is a homeomorphism, then
$f_\# : C_n(X) \to C_n(Y)$ is an isomorphism
and $f_* : H_n(X) \to H_n(Y)$ is an isomorphism

Thus singular homology is a topological invariant.

Prop 2.6: Suppose X_α are the path components of X. Then $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$ and $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$

Prop 2.7: If X is non-empty, path-connected, then $H_0(X) = \mathbb{Z}$.

Prop 2.8: $H_n(point) = 0$ for $n > 0$ and $H_0(point) = \mathbb{Z}$.

Reduced homology

The reduced homology groups $\widetilde{H}_n(X)$ are the homology groups of the augmented chain complex:

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$. Note $\epsilon\partial_1 = 0$.

Thus ϵ induces a map $H_0 = C_0/Im(\partial_1) \to \mathbb{Z}$ w/ kernel $\widetilde{H}_0(X)$.

Thus $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$. since ϵ onto

For $n > 0$, $\widetilde{H}_n(X) = ker(\partial_n)/Im(\partial_{n+1}) = H_n(X)$.

Hence $\widetilde{H}_n(point) = 0$ for all n.

Mobius band = Cross cap

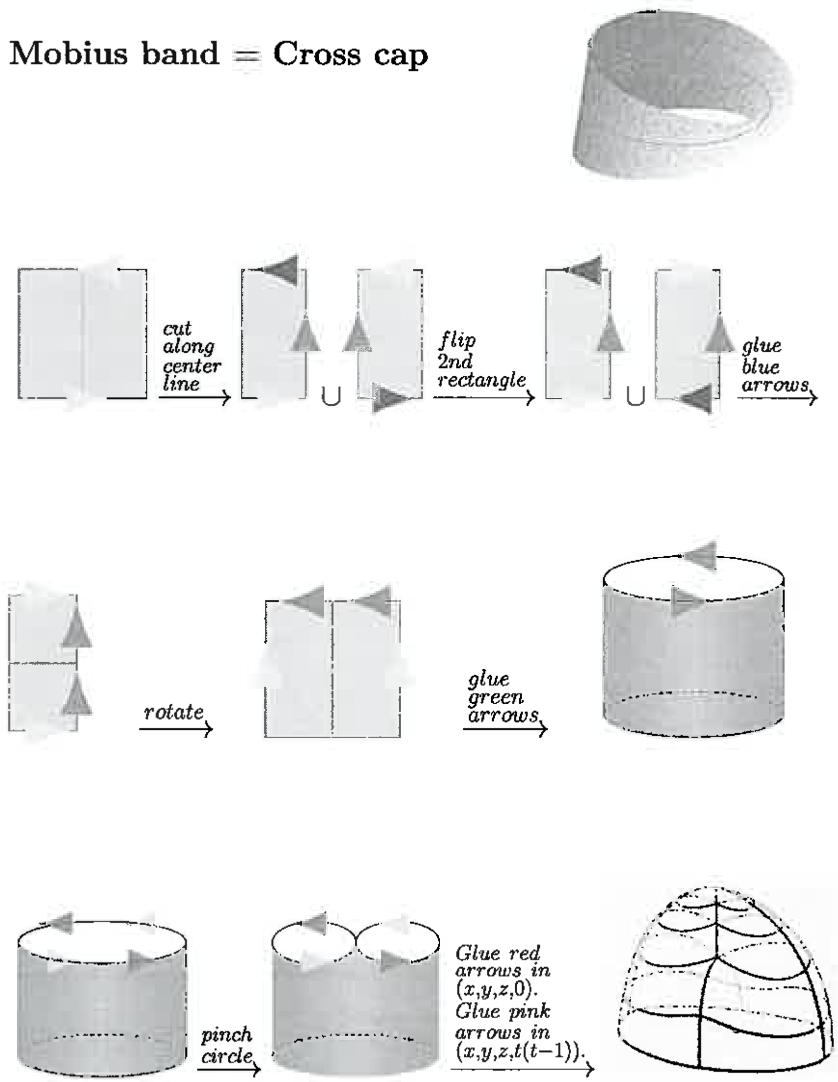


Figure 1: Cross cap in $\mathbb{R}^4$. Last figure from http://www.freud-lacan.com/freud/Champs_specialises/Langues_etrangeres/Anglais/Le_cross_cap_de_Lacan_ou_asphere

A chain complex is a sequence of homomorphisms of abelian groups such that $\partial_n\partial_{n+1} = 0$ for all n:

$$\dots \xrightarrow{\partial_{n+2}} G_{n+1} \xrightarrow{\partial_{n+1}} G_n \xrightarrow{\partial_n} G_{n-1} \xrightarrow{\partial_{n-1}} \dots.$$

Given a chain complex, define homology $H_n = Ker(\partial_n)/Im(\partial_{n+1})$.

A *chain map* $\phi : (C_\bullet, \partial_\bullet) \to \left(\widetilde{C}_\bullet, \widetilde{\partial}_\bullet\right)$ is a collection of homomorphisms $\phi_n : C_n \to \widetilde{C}_n$ such that the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\phi_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\phi_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\phi_{n-1}"] & \cdots \\
\cdots \arrow[r] & \widetilde{C}_{n+1} \arrow[r, "\widetilde{\partial}_{n+1}"] & \widetilde{C}_n \arrow[r, "\widetilde{\partial}_n"] & \widetilde{C}_{n-1} \arrow[r, "\widetilde{\partial}_{n-1}"] & \cdots
\end{tikzcd}$$

A chain map $\phi : (C_\bullet, \partial_\bullet) \to \left(\widetilde{C}_\bullet, \widetilde{\partial}_\bullet\right)$ induces a map on homology $\phi_* : H_n \to \widetilde{H}_n$.

Suppose $f : X \to Y$ is continuous. singular homology

f induces the the chain map $f_\# : (C_\bullet(X), \partial_\bullet) \to (C_\bullet(Y), \partial_\bullet)$.

$f_\#(\sigma : \Delta \to X) = f \circ \sigma : \Delta \to Y$ and extend linearly.

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4 methods for calculating homology:

1.) via definition 2.) via matrices 3.) hand-waving

4.) $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ for X path connected. That is H_1 is the abelianization of π_1 (See Appendix 2A Hatcher).