

If $H < G = \pi_1(X, x_0)$, $\exists p: \tilde{X} \rightarrow X$ such that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$

Step 1: Define \tilde{X} by

1.) $P = \{[\alpha] \mid \alpha \text{ a path in } X \text{ starting at } x_0\}$

2.) $\tilde{X} = P / \sim$ where $\alpha \sim \beta$ iff $\alpha\beta^{-1} \in H$.

Let $\alpha^\#$ denote the equivalence class of $\alpha \in P$.

Note: If $[\alpha] = [\beta]$, then $\alpha^\# = \beta^\#$ since $\alpha\beta^{-1} = e \in H$.

Note: If $\alpha^\# = \beta^\#$, then $(\alpha\delta)^\# = (\beta\delta)^\#$ when the product is defined since $(\alpha\delta)(\beta\delta)^{-1} = \alpha\delta\delta^{-1}\beta^{-1} = \alpha\beta^{-1} \in H$.

Define $p: \tilde{X} \rightarrow X$, $p(\alpha^\#) = \alpha(1)$.

Note p is onto since X is path connected.

Step 2: Topologize \tilde{X}

Method 1: Give P the compact-open topology

$S(C, U) = \{\alpha \mid \alpha: [0, 1] \rightarrow X \text{ such that } \alpha(C) \subset U\}$

$\mathcal{S} = \{S(C, U) \mid C \text{ compact in } [0, 1], U \text{ open in } X\}$ is subbases for compact-open topology on the set of paths P .

Give $\tilde{X} = P / \sim$ the quotient topology.

Or equivalently,

$$B(U, \alpha) = B(U, \alpha\delta) \text{ where } \alpha\delta\beta^{-1} \in H$$

$B(U, \alpha)$ only depends on U and α doesn't depend on which $\beta^\# \in (U, \alpha)$

Method 2: Let $\alpha \in P$, U path connected open nbhd of $\alpha(1)$.

$B(U, \alpha) = \{(\alpha * \delta)^\# \mid \delta \text{ path in } U \text{ such that } \delta(0) = \alpha(1)\}$

Claim: $\{B(U, \alpha) \mid \alpha \in P, U \text{ path connected nbhd of } \alpha(1)\}$ is a basis for a topology on \tilde{X} .

1.) $\alpha^\# = (\alpha * e_{\alpha(1)})^\# \in B(U, \alpha)$

2.) Suppose $\beta^\# \in B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$.

$\beta^\# \in B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$ where V is path connected component of $U_1 \cap U_2$ containing $\beta(1)$.



$S \subset V = U_1 \cap U_2$
 $\beta \in B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$

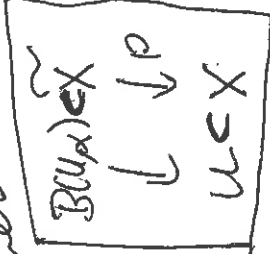
$\beta^\# = (\alpha_1\delta_1)^\# = (\alpha_2\delta_2)^\#$ implies $(\beta\delta)^\# = (\alpha_1\delta_1\delta)^\# = (\alpha_2\delta_2\delta)^\#$

Claim: $\beta^\# \in B(U, \alpha)$ implies $B(U, \alpha) = B(U, \beta)$.

$\beta^\# \in B(U, \alpha)$ implies $\beta^\# = (\alpha\delta)^\#$ implies $\alpha^\# = (\beta\delta^{-1})^\#$. Thus $\alpha^\# \in B(U, \beta)$.

$(\beta\delta)^\# = (\alpha\delta\delta)^\# \in B(U, \alpha)$. Thus $B(U, \beta) \subset B(U, \alpha)$. Similarly, $B(U, \alpha) \subset B(U, \beta)$.

cont & open operation
 \hookrightarrow So if b_{ij} p homeo



Claim: $p|_{B(U, \alpha)} : B(U, \alpha) \rightarrow U$ is a bijection.

Onto: Recall $p(B(U, \alpha)) = U$.

1.1: Suppose $p((\alpha\delta_1)^\#) = p((\alpha\delta_2)^\#)$ where $\delta_1, \delta_2 \subset U$.
 Then $\delta_1(1) = \delta_2(1)$.



Recall $i_* : \pi_1(U, z) \rightarrow \pi_1(X, z)$ is trivial.

Thus $\delta_1\delta_2^{-1} = e$ in $\pi_1(X, z)$.

Thus $[\alpha\delta_1] = [\alpha\delta_2]$ and $(\alpha\delta_1)^\# = (\alpha\delta_2)^\#$.

Step 5, 6. \tilde{X} is path connected.

Let $\tilde{x} = e_{x_0}$, the constant path at x_0 . Then $p(\tilde{x}) = e_{x_0}(1) = x_0$.
 Let $\alpha^\# \in \tilde{X}$. \hookrightarrow Find path from \tilde{x} to $\alpha^\#$ in \tilde{X} .
 Let $c \in [0, 1]$. Define $\alpha_c : I \rightarrow X, \alpha_c(t) = \alpha(tc)$.
 Thus α_c is a path from x_0 to $\alpha(c)$. $\alpha_c^\# \in \tilde{X}$.
 Define $\tilde{\alpha} : I \rightarrow \tilde{X}$ by $\tilde{\alpha}(c) = (\alpha_c)^\#$.
 $p(\tilde{\alpha}(c)) = p((\alpha_c)^\#) = \alpha_c(1) = \alpha(c)$.
 Hence $p \circ \tilde{\alpha} = \alpha$.

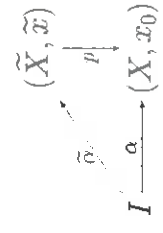
Thus $\tilde{\alpha}$ is the lift of α starting at x_0 if $\tilde{\alpha}$ is continuous.

Define $\tilde{\alpha} : I \rightarrow \tilde{X}$ by $\tilde{\alpha}(c) = (\alpha_c)^\#$.

$p(\tilde{\alpha}(c)) = p((\alpha_c)^\#) = \alpha_c(1) = \alpha(c)$.

Hence $p \circ \tilde{\alpha} = \alpha$.

Thus $\tilde{\alpha}$ is the lift of α starting at x_0 if $\tilde{\alpha}$ is continuous.



constant path at x_0

Thus $B(U, \alpha) \cap B(U, \beta) \neq \emptyset$ implies $B(U, \alpha) = B(U, \beta)$.

Step 3. p is open and continuous.

Note $p(B(U, \alpha)) = U$ since $p((\alpha\delta)^\#) = \delta(1)$ and U is path connected. Thus p is open.

Claim p continuous. Let W be open in X . Let $\alpha^\# \in p^{-1}(W)$. Let U be a path connected component of W such that $\alpha(1) \in U$. Note U is open. Then $\alpha^\# \in B(U, \alpha) \subset p^{-1}(W)$.

Step 4. Claim: $\forall z \in X, \exists U$ open in X such that U is evenly covered by p .

Let $z \in X$. Take U path connected nbhd of z such that

$i_* : \pi_1(U, z) \rightarrow \pi_1(X, z)$ is trivial.

Note U exists since X is path connected, locally path connected, and semilocally simply connected.

can get disjoint union by removing repeats

Claim: $p^{-1}(U) \cong \bigcup_{\alpha \text{ path from } x_0 \text{ to } z} B(U, \alpha) \cong U$.

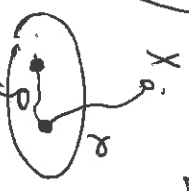
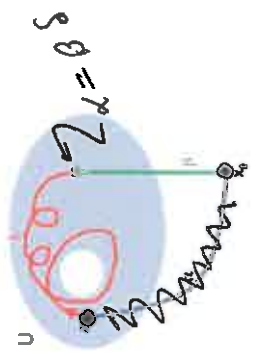
(\supset): Clear since $p(B(U, \alpha)) \cong U$.

(\subset): If $\beta^\# \in p^{-1}(U)$, then $\beta(1) \in U$.

Let δ be a path in U from z to $\beta(1)$.

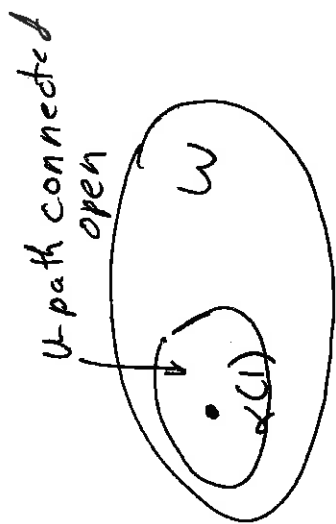
Let $\alpha = \beta\delta^{-1}$ is a path from x_0 to z .

Thus $\beta^\# = (\alpha\delta)^\# \in B(U, \alpha)$.



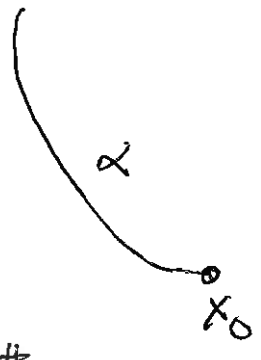
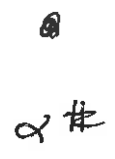


$\sim X$



$$p(B(U, \alpha)) = U$$

$$B(U, \alpha) \subset p^{-1}(w)$$



$$\alpha_c^\# \in B(\alpha, \beta) = B(\alpha, \alpha_c)$$

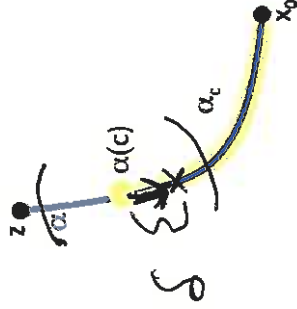
Claim: $\tilde{\alpha} : I \rightarrow \tilde{X}$ is continuous. Let $c \in I$.

$B(U, \alpha_c)$ is an arbitrary basis element containing $\tilde{\alpha}(c) = \alpha_c^\#$.

$\alpha : I \rightarrow X$ is continuous.

Thus $\exists \varepsilon > 0$ such that if $t \in (c - \varepsilon, c + \varepsilon)$, then $\alpha(t) \in U$.

Then $\alpha_t^\# = (\alpha_c \delta)^\# \in B(U, \alpha_c)$



$$[\alpha] \in p_*(\pi_1(\tilde{X}, \tilde{x}))$$

*Note compared to the standard $\delta - \varepsilon$ definition of continuity, our $\varepsilon = \text{old } \delta$ and our U relates to the old ε .

Step 7. Claim $p_*(\pi_1(\tilde{X}, \tilde{x})) = H$.

$[\alpha] \in p_*(\pi_1(\tilde{X}, \tilde{x}))$ iff $\tilde{\alpha} \in \pi_1(\tilde{X}, \tilde{x})$

iff $\tilde{\alpha}$ is a loop in \tilde{X} based at \tilde{x}

iff $\alpha^\# = \tilde{\alpha}(1) = \tilde{x}$ iff $[\alpha] = [\alpha * e_{x_0}^{-1}] \in H$.

Thus $p_*(\pi_1(\tilde{X}, \tilde{x})) = H$