Pointed

PROPOSITION 79.2

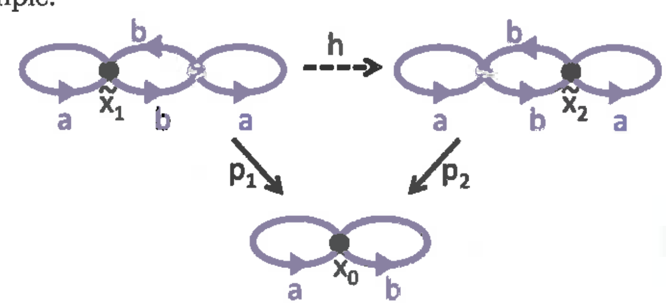
$$\begin{tikzcd}(\widetilde{X}_1, \widetilde{x}_1) \arrow[dr, "p_1"'] \arrow[rr, dashed, "\text{then } \exists! h"] & & (\widetilde{X}_2, \widetilde{x}_2) \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd}$$

$p_1 \colon \widetilde{X}_1 \to X$ and $p_2 \colon \widetilde{X}_2 \to X$ are equivalent via a homeomorphism $h \colon \widetilde{X}_1 \to \widetilde{X}_2$ taking a basepoint $\widetilde{x}_1 \in p_1^{-1}(x_0)$ to a basepoint $\widetilde{x}_2 \in p_2^{-1}(x_0)$

if and only if = , not just isomorphic

$$(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2)).$$

Example:

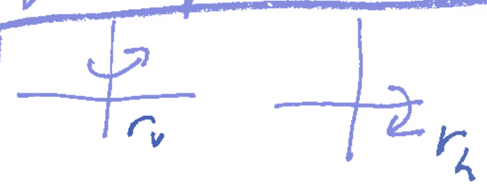


$$(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = < a, b^{2}, bab^{-1} > = (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2)).$$

$h = id$ ← not pointed

$h = r_h \circ r_v$ ← pointed

Prop 79.2



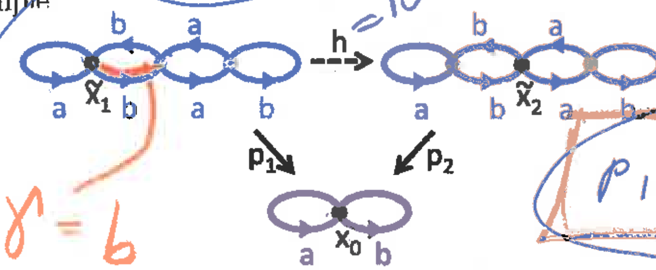
PROPOSITION 79.3a Given covering map $\begin{matrix}\widetilde{X} \\ p\downarrow \\ X\end{matrix}$ and $\widetilde{x}_1, \widetilde{x}_2 \in p_1^{-1}(x_0)$,

$p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$ and $p_*(\pi_1(\widetilde{X}, \widetilde{x}_2))$are conjugate in $\pi_1(X, x_0)$.

Moreover, let $H_1 = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$ and $H_2 = p_*(\pi_1(\widetilde{X}, \widetilde{x}_2))$, let γ be a path in $\widetilde{X}$ from $\widetilde{x}_1$ to $\widetilde{x}_2$, and let $\alpha = p \circ \gamma \in \pi_1(X, x_0)$ then $H_1 = \alpha H_2 \alpha^{-1}$

$H_1 \neq H_2$

Example:



$p_1 = p_2$

$H_1 = (p_1)_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = < a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} > \subset \pi_1(X, x_0) = \langle a, b \rangle$

$H_2 = (p_2)_*(\pi_1(\widetilde{X}, \widetilde{x}_2)) = < bab^{-1}, b^2, a^2, aba^{-1} >$

$bH_2b^{-1} = < bbab^{-1}b^{-1}, bb^2b^{-1}, ba^2b^{-1}, baba^{-1}b^{-1} >$ ← loops at $\tilde{x}_1$

$= < b^2ab^{-2}, b^2, ba^2b^{-1}, baba^{-1}b^{-1} >$

$a, b^2 \in H_1$ implies $b^2ab^{-2} \in H_1$

$b^2, b^2ab^{-2} \in bH_2b^{-1}$ implies $b^{-2}b^2ab^{-2}b^2 = a \in bH_2b^{-1}$

Thus $\boxed{H_1 = bH_2b^{-1}}$

Not pointed $id(\tilde{x}_1) \neq \tilde{x}_2$

79.3b

Proposition 3. Given covering map $\begin{array}{c}(\widetilde{X},\widetilde{x}_0)\\ \downarrow p\\ (X,x_0)\end{array}$, $H_0 = p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$.

If H is a subgroup of $\pi_1(X,x_0)$, such that $H_0 = \alpha H \alpha^{-1}$, then

$\exists \widetilde{x}_1 \in p^{-1}(x_0)$ such that $H = (p)_*(\pi_1(\widetilde{X}_1,\widetilde{x}_1))$.

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combine 79.2 & 79.3

Proposition ~~4~~. 79.4

$$\begin{tikzcd} \widetilde{X}_1 \arrow[dr, "p_1"'] \arrow[rr, dashed, "\text{then } \exists h"] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

not pointed

Suppose $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2) = x_0$ The covering maps p_1, and p_2 are equivalent iff the subgroups $\underbrace{(p_1)_*(\pi_1(\widetilde{X}_1,\widetilde{x}_1))}_{H_1}$ and $\underbrace{(p_2)_*(\pi_1(\widetilde{X}_2,\widetilde{x}_2))}_{H_2}$ are conjugate in $\pi_1(X,x_0)$.

Subgroups of $\mathbb{Z}$ are $\{e\}$, and $n\mathbb{Z}$, $n = 1,2,3,\ldots$

$\begin{array}{c}\mathbb{R}\\ \downarrow p\\ S^1\end{array}$ $\pi_1(\mathbb{R},0) = \{e\}$. Thus $p_*(\pi_1(\mathbb{R},0)) = \{e\}$

$\begin{array}{c}S^1\\ \downarrow p_n = z^n\\ S^1\end{array}$ $\pi_1(S^1,0) = \mathbb{Z}$ and $p_{n*}(\pi_1(S^1,0)) = n\mathbb{Z}$

By the above proposition, any covering map of S^1 is equivalent to one of the covering maps above.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[r, "q"] & Y \\ Z \arrow[ur, "r"'] & \end{tikzcd}$$

Proposition 5.

Let p, q, r be continuous maps with $p = r \circ q$. Then

i.) If q and r are covering maps and if either (Z has a universal covering space) or ($r^{-1}(z)$ is finite for all $z \in Z$), then p is a covering map.

ii.) If p and r are covering maps, then q is a covering map.

iii.) If p and q are covering maps, then r is a covering map.

$$\begin{tikzcd} \widetilde{\widetilde{X}} \arrow[d, "p"'] \arrow[r, "q"] & \widetilde{X} \\ X \arrow[ur, "r"'] & \end{tikzcd}$$

Proposition 6. If p is a covering map and if $\pi_1(\widetilde{X}) = \{e\}$, then given any covering map r, there exists covering map q such that $p = r \circ q$.

Definition 0.1. If $\begin{array}{c}\widetilde{\widetilde{X}}\\ \downarrow p\\ X\end{array}$ is a covering map and if $\pi_1(\widetilde{\widetilde{X}}) = \{e\}$,

then $\widetilde{\widetilde{X}}$ is called the *universal covering space of* X.