$$
\begin{tikzcd}
 & \pi_1(U) \ni [g]_U = b \arrow[d, "i_{U*}"] \\
{[g]_{U\cap V}} \in \pi_1(U \cap V) \arrow[ur, "i_1"] \arrow[r, "inclusion"] \arrow[dr, "i_2"'] & \pi_1(X) \ni [g]_X = e \\
 & \pi_1(V) \ni [g]_V \quad e \arrow[u, "i_{V*}"']
\end{tikzcd}
$$

$\| \; b$

$$i_{U*}(i_1([g]_{U\cap V})) = i_{U*}([g]_U) = [g]_X = i_{U\cap V*}([g]_{U\cap V})$$

$$i_{V*}(i_2([g]_{U\cap V})) = i_{V*}([g]_V) = [g]_X = i_{U\cap V*}([g]_{U\cap V})$$

$\pi_1(U) = \langle a, b \rangle$	$[g]_U = b$	
$\pi_1(V) = \{e\}$	$[g]_V = e$	$[g]_{U \cup V} = e$
$\pi_1(U\cap V) = \langle b \rangle$	$[g]_{U\cap V} = b$	

$G_1 * G_2$

$\langle a_1, \dots, a_i, b_1, \dots, b_j \mid r_1, \dots, r_e, s_1, \dots, s_m \rangle$

By group theory, given homomorphism $\phi_1 : G_1 \to H$, then there exists a unique homomorphism $\Phi_i : G_1 * G_2 \to H$ such that the following diagram commutes.

$$
\begin{tikzcd}
G_1 \arrow[d, "i_1"] \arrow[dr, "\phi_1"] & \\
G_1 * G_2 \arrow[r, dashed, "\text{then } \exists!\Phi"] & H \\
G_2 \arrow[u, "i_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}
$$

Φ is defined by defining it on its generators:

$$\Phi(g_i) = \phi_i(g_i) \text{ where } g_i \in G_i,\ i = 1 \text{ or } 2.$$

We extend to arbitrary words in $G_1 * G_2$ using the definition of group homomorphism:

$$\Phi(g_1 g_2 \cdots g_n) = \phi_{i_1}(g_1)\phi_{i_2}(g_2)\cdots\phi_{i_n}(g_n) \text{ where } i_k = \begin{cases} 1 & g_k \in G_1 \\ 2 & g_k \in G_2 \end{cases}$$

Since ϕ_i are homomorphisms, if $w_1 = w_2$ are two equivalent words in $G_1 * G_2$, then $\Phi(w_1) = \Phi(w_2)$.

Thus Φ is a well-defined homomorphism.

Thus we have the following:

$$\begin{tikzcd}
\pi_1(U) \arrow[d, "i_{U*}"'] \arrow[dr, "if\ \exists\phi_1"] & \\
\pi_1(U) * \pi_1(V) \arrow[r, dashed, "\text{then } \exists!\Phi"] & H \\
\pi_1(V) \arrow[u, "i_{V*}"] \arrow[ur, "if\ \exists\phi_2"'] &
\end{tikzcd}$$

We also know from group theory that if N is any normal subgroup in $G_1 * G_2$, then $\boxed{\textbf{IF } \Phi \text{ exists}}$ for the following diagram, then Φ is ! since $\Phi([g_i]) = \phi_i(g_i)$ where $g_i \in G_i$, $i = 1$ or 2

$$\begin{tikzcd}
G_1 \arrow[d, "i_1"'] \arrow[dr, "\phi_1"] & \\
G_1 * G_2/N \arrow[r, "\text{If } \exists\Phi\text{, then } \Phi \text{ is !}"] & H \\
G_2 \arrow[u, "i_2"] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

$$\pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$$

Thus if Φ exists, then Φ is unique. $\Leftarrow$ part of Thm 70.1

$$\begin{tikzcd}
\pi_1(U) \arrow[d, "i_{U*}"'] \arrow[dr, "if\ \exists\phi_1"] & \\
\pi_1(U) * \pi_1(V)/ker(j) \arrow[r, "\text{If } \exists\Phi\text{, then } \Phi \text{ is !}"] & H \\
\pi_1(V) \arrow[u, "i_{V*}"] \arrow[ur, "if\ \exists\phi_2"'] &
\end{tikzcd}$$

We know how to define Φ **IF** Φ exists:

$$\Phi([g_i]) = \phi_i(g_i) \text{ where } g_i \in G_i,\ i = 1 \text{ or } 2$$

To determine if this Φ is well-defined, we only need to check if $\Phi(N) = \{e\}$.

$$\begin{tikzcd}
G_1 \arrow[d, "i_1"'] \arrow[dr, "\phi_1"] & \\
G_1 * G_2/N \arrow[r, dashed, "\text{If } \exists\Phi\text{, then } \Phi \text{ is !}"] & H \\
G_2 \arrow[u, "i_2"] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

$G = \pi_1(U) * \pi_1(V)$

Let N = least normal subgroup generated by

$$\{i_1(c_1)^{-1}i_2(c_1), ..., i_1(c_n)^{-1}i_2(c_n)\}$$

I.e., N is generated by $\{gd_1g^{-1}, ..., gd_ng^{-1} \mid g \in G\}$
where $d_k = i_1(c_k)^{-1}i_2(c_k)$

and where the c_i's are the generators of $\pi(U \cap V)$

So that Φ exists, we need to expand our commutative diagram to include i_1 and i_2 as below:

$$\begin{tikzcd}
 & \pi_1(U) \arrow[d, "i_{U*}"] \arrow[dr, "if\ \exists\phi_1"] & \\
\pi_1(U \cap V) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] \arrow[r, "inclusion"] & \boxed{\pi_1(U) * \pi_1(V)/N} \arrow[r, dashed, "then\ \exists!\Phi"] & H \\
 & \pi_1(V) \arrow[u, "i_{V*}"'] \arrow[ur, "if\ \exists\phi_2"'] &
\end{tikzcd}$$

$\Phi(i_1(c_k)^{-1}i_2(c_k)) = [\Phi(i_1(c_k))]^{-1}\ \Phi(i_2(c_k))$

$= [\phi_1(i_1(c_k))]^{-1}\ \phi_2(i_2(c_k)) = [\phi_1(i_1(c_k))]^{-1}\ \phi_1(i_1(c_k)) = e$

since by hypothesis the maps commute

$\Phi(gd_kg^{-1}) = \Phi(g)\Phi(d_k)\Phi(g^{-1}) = \Phi(g)\Phi(g^{-1}) = e$

Since Φ sends the generators of N to e, $\Phi(N) = \{e\}$.

Thus Thm 70.2 $[\pi(X) = \pi_1(U) * \pi_1(V)/N]$ implies Thm 70.1.

Thm 70.1 implies Thm 70.2:

$$\begin{tikzcd}
 & \pi_1(U) \arrow[d, "i_{U*}"] \arrow[dr, "if\ \exists\phi_1"] & \\
\pi_1(U \cap V) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] \arrow[r, "inclusion"] & \pi_1(X) \arrow[r, dashed, "then\ \exists!\Phi"] & \pi_1(U) * \pi_1(V)/N \\
 & \pi_1(V) \arrow[u, "i_{V*}"'] \arrow[ur, "if\ \exists\phi_2"'] &
\end{tikzcd}$$

ϕ_1 = inclusion

ϕ_2 = inclusion

commutes since N

$j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ induced by the two inclusion maps is surjective.

I.e., $\pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$

Claim: $N \subset ker(j)$.

by first page

$j(i_1(c_k)^{-1}i_2(c_k)) = [j(i_1(c_k))]^{-1} * j(i_2(c_k)) = [i_{U\cap V}(c_k)]^{-1}i_{U\cap V}(c_k)$
$= e$

$\exists\ \Phi$ by Thm 70.1

Thus j induces a map

$$k : \pi_1(U) * \pi_1(V)/N \twoheadrightarrow \pi_1(U) * \pi_1(V)/ker(j) = \pi_1(X)$$

since if $N \subset M$ are normal subgroups of G, then
$i : G/N \twoheadrightarrow G/M$, $i(gN) = gM$ is a homomorphism.

By taking ϕ_i to be inclusion maps in Thm 70.1, $\exists\ \Phi = k^{-1}$.

Thus k is an isomorphism.

$$\langle \underbrace{a_1, \ldots, a_i, b_1, \ldots, b_j \mid r_1, \ldots, r_\ell, s_1, \ldots, s_m}_{\pi_1(U) * \pi_1(V)}, \underbrace{i_1(c_1)\, i_2(c_1)^{-1}, \ldots, i_1(c_n)\, i_2(c_n)^{-1}}\rangle$$

$j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ induced by the two inclusion maps is surjective.

$$\text{Thus } \pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$$

Thm 70.1: U, V, $U \cap V$ open and path-connected.

$$\begin{tikzcd}
 & \pi_1(U) \arrow[d, "i_{U*}"] \arrow[dr, "if\ \exists\phi_1"] & \\
\pi_1(U \cap V) \arrow[ur, "i_1"] \arrow[r, "inclusion"] \arrow[dr, "i_2"'] & \pi_1(X) \arrow[r, dashed, "\text{then } \exists!\Phi"] & H \\
 & \pi_1(V) \arrow[u, "i_{V*}"'] \arrow[ur, "if\ \exists\phi_2"'] &
\end{tikzcd}$$

We need to show that Φ is well defined.
Note the definition of Φ is "obvious."

Recall that if $g \in \pi_1(X)$,
then $g : I = \cup[t_j, t_{j+1}] \to X$ is a loop in $X = U \cup V$.

We can take $g_j = g|_{[t_{j-1},t_j]}$ to be paths in either U or V ← by Lebesgue # lemma

For each t_j, choose paths α_j from x_0 to $g(t_j)$ such that

$$\circledast \begin{cases} \text{If } g(t_j) \in U \cap V, & \text{choose } \alpha_j \in U \cap V \\ \text{If } g(t_j) \in U - V, & \text{choose } \alpha_j \in U \\ \text{If } g(t_j) \in V - U, & \text{choose } \alpha_j \in V \end{cases}$$

Then $g = (\alpha_0 * g_1 * {\alpha_1}^{-1})(\alpha_1 * g_2 * {\alpha_2}^{-1}) \cdots (\alpha_{n-1} * g_n * {\alpha_n}^{-1})$
where for each j, $(\alpha_{j-1} * g_j * {\alpha_j}^{-1})$ is in $\pi_1(U)$ or in $\pi_1(V)$.

$\Phi(g) = \phi_{i_1}(\alpha_0 g_1 \alpha_1^{-1}) \phi_{i_2}(\alpha_1 g_2 \alpha_2^{-1}) \cdots \phi_{i_n}(\alpha_{n-1} g_n \alpha_n^{-1})$

where $i_k = \begin{cases} 1 & \text{if } g_k \subset U \\ 2 & \text{if } g_k \subset V \end{cases}$

↖ could depend on choice of α_i's

Claim: Given $g = g_1 \cdots g_n \sim f_1 \cdots f_l$, with specified paths α_j from the basepoint x_0 to $g(t_j)$ and paths β_j from x_0 to $f(s_j)$ satisfying conditions $\circledast$, then

$$\Phi(g_1 \cdots g_n) = \Phi(f_1 \cdots f_l).$$

Sub-Claim: If we subdivided the path g_j into h_1 and h_2, then we can replace g_j with $h_1 h_2$.

I.e., $\Phi(g_1 \cdots g_n) = \Phi(g_1 \cdots g_{j-1} h_1 h_2 g_{j+1} \cdots g_n)$

WLOG $g_j : [t_{j-1}, t_j] \to U$. Let $y \in [t_{j-1}, t_j]$ such that

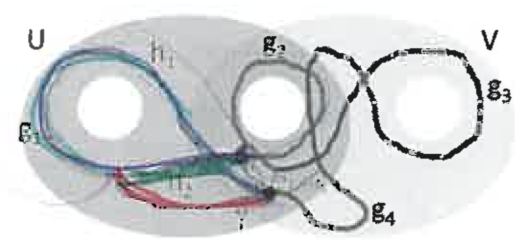
$$h_1 = g_j|_{[t_{j-1},y]} \text{ and } h_2 = g_j|_{[y,t_j]}.$$

Let γ be any path from x_0 to y satisfying conditions $\circledast$.

Then $\phi_1(\alpha_{j-1} * g_j * {\alpha_j}^{-1}) = \phi_1(\alpha_{j-1} * h_1 h_2 * {\alpha_j}^{-1})$

$= \phi_1(\alpha_{j-1} * h_1 \gamma^{-1} \gamma h_2 * {\alpha_j}^{-1}) = \phi_1(\alpha_{j-1} * h_1 \gamma^{-1}) \phi_1(\gamma h_2 * {\alpha_j}^{-1})$

Thus $\Phi(g_1 \cdots g_n) = \Phi(g_1 \cdots g_{j-1} h_1 h_2 g_{j+1} \cdots g_n)$ with the associated paths α_i and γ.



$g = (\alpha_0 h_1 h_2 \overline{\alpha}_1)(\alpha_1 g_2 \overline{\alpha}_2)(\alpha_2 g_3 \overline{\alpha}_3)(\alpha_3 g_4 \overline{\alpha}_4)$
where α_0 and α_4 are constant maps.