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For 59.1

$[\,|\,|\,\underset{a_i}{\bullet}\,|\,*\,|\,]$ want $g(a_i) \in U \cap V$ p.c

Proof of thm 59.1:

Let $g : I \to X$ be a loop in $X = U \cup V$.

Since U and V are open, $I \subset g^{-1}(U) \cup g^{-1}(V)$.

By the Lebesgue number lemma, $\exists\ \delta > 0$ such that if $A \subset X$ with $\text{diam}(A) < \delta$, then $A \subset g^{-1}(U)$ or $A \subset g^{-1}(V)$.

Thus if $I = \cup[a_i, a_{i+1}]$ where $\text{diam}([a_i, a_{i+1}]) < \delta$,
then $[a_i, a_{i+1}] \subset g^{-1}(U)$ or $[a_i, a_{i+1}] \subset g^{-1}(V)$.

Thus
$g([a_i, a_{i+1}]) \subset g(g^{-1}(U)) \subset U$ or $g([a_i, a_{i+1}]) \subset g(g^{-1}(V)) \subset V$.

If there exists a_k such that $g(a_k) \notin U \cap V$,
replace the intervals $[a_{k-1}, a_k]$ and $[a_k, a_{k+1}]$ with $[a_{k-1}, a_{k+1}]$.

If $g(a_k) \in U$, then $g(a_k) \notin V$. Hence
$g([a_{k-1}, a_k]) \subset U$ & $g([a_k, a_{k+1}]) \subset U$. Thus $g([a_{k-1}, a_{k+1}]) \subset U$.

Similary, $g(a_k) \in V$ implies $g([a_{k-1}, a_{k+1}]) \subset V$.

Thus we can write $I = \cup[c_i, c_{i+1}]$ where
$c_i \in U \cap V\ \forall i$ and $g([c_i, c_{i+1}]) \subset U$ or $g([c_i, c_{i+1}]) \subset V$

Since $U \cap V$ is path connected, $\exists$ a path α_i between x_0 and c_i.

Thus $g = (\alpha_0 * g_1 * \overline{\alpha_1})(\alpha_1 * g_2 * \overline{\alpha_2}) \cdots (\alpha_{n-1} * g_n * \overline{\alpha_n})$

where for each i, $(\alpha_{i-1} * g_i * \overline{\alpha_i})$ is in $\pi_1(U)$ or in $\pi_1(V)$.

I.e., $j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ induced by the two inclusion maps is surjective.

← only need $U \cap V$ p.c.

$j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ induced by the two inclusion maps is surjective.

Thus $\pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$

SVK

Thm 70.1: $U, V, U \cap V$ open and path-connected.

hyp

diagram commuting

conclusion

$\exists!\,\Phi$

$$\begin{tikzcd}
& \pi_1(U) \arrow[d, "i_{U*}"] \arrow[dr, "if\ \exists\phi_1"] & \\
\pi_1(U \cap V) \arrow[ur, "i_1"] \arrow[r, "inclusion"] \arrow[dr, "i_2"'] & \pi_1(X) \arrow[r, "then\ \exists!\Phi"] & H \\
& \pi_1(V) \arrow[u, "i_{V*}"'] \arrow[ur, "if\ \exists\phi_2"'] &
\end{tikzcd}$$

$[g]_U$

$[g]_{U\cap V}$

$$\begin{tikzcd}
& \pi_1(U) \arrow[d, "i_{U*}"] \\
\pi_1(U \cap V) \arrow[ur, "i_1"] \arrow[r, "inclusion"] \arrow[dr, "i_2"'] & \pi_1(X) \\
& \pi_1(V) \arrow[u, "i_{V*}"']
\end{tikzcd}$$

$[g]_{U\cup V}$

get this for free

$$\begin{tikzcd}
\pi_1(U) \arrow[d, "i_{U*}"] \arrow[dr, "if\ \exists\phi_1"] & \\
\pi_1(U) * \pi_1(V) \arrow[r, "then\ \exists!\Phi"] & H \\
\pi_1(V) \arrow[u, "i_{V*}"'] \arrow[ur, "if\ \exists\phi_2"'] &
\end{tikzcd}$$

! since maps commute

If $g = \tilde{g}_1 * \tilde{g}_2 * \cdots * g_n = \phi_1(\tilde{g}_1) * \cdots$

. . . if $\tilde{g}_1 \in \pi_1(U)$

WLOG $g_1 \in U$

~~where $i \neq 0$~~