$\exists \delta$ ↑ Lebesgue #

Lemma 27.5 (*The Lebesgue number lemma*)
If $\mathcal{U}$ is an open covering of the compact metric space X, then $\exists \delta > 0$ such that if $A \subset X$ with $\text{diam}(A) < \delta$, then $\exists\, U \in \mathcal{U}$ such that $A \subset U$.

Thm 59.1: Suppose $X = U \cup V$ where U, V are open and $\boxed{U \cap V \text{ is path connected.}}$ Let $i_U : U \to X$ and $i_V : V \to X$ be inclusion maps. Then $\pi_1(X)$ is generated by the images of i_{U*} and i_{V*}.

I.e., If $g \in \pi_1(X)$, the $g = g_1 * g_2 * \ldots * g_n$ where for each i, g_i is in either $i_{U*}(\pi_1(U))$ or $i_{V*}(\pi_1(V))$.

I.e., $j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ induced by the two inclusion maps is surjective. ← Thm 59.1

I.e, $\pi_1(X) = \pi_1(U) * \pi_1(V)/ker(j)$

$= < a_1, ..., a_i, b_1, ..., b_j \mid s_1, ..., s_l, t_1, ..., t_m > /ker(j)$ $\blacksquare$

Theorem 70.2. $\ker(j)$ = least normal subgroup generated by $\{i_U(c_1)^{-1} i_V(c_1), ..., i_U(c_n)^{-1} i_V(c_n)\}$. $\blacksquare$

I.e., $\pi_1(X) = < a_1, ..., a_i, b_1, ..., b_j \mid s_1, ..., s_l, t_1, ..., t_m,$ $\blacksquare$
$i_U(c_1)^{-1} i_V(c_1), ..., i_U(c_n)^{-1} i_V(c_n) >$

The following maps are all induced by inclusion

$$\begin{tikzcd}
 & \pi_1(U) \arrow[d, "i_{U*}"] \\
\pi_1(U \cap V) \arrow[ur, "i_1"] \arrow[r, "inclusion"] \arrow[dr, "i_2"'] & \pi_1(X) \\
 & \pi_1(V) \arrow[u, "i_{V*}"']
\end{tikzcd}$$

Thm 70.1: U, V, $U \cap V$ open and path-connected.

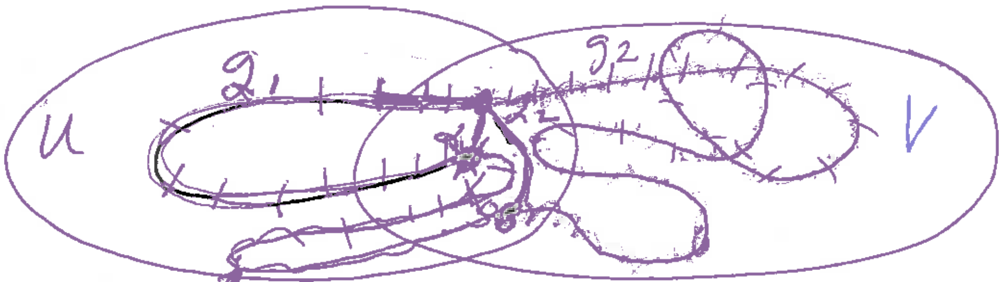
$$\begin{tikzcd}
 & \pi_1(U) \arrow[d, "i_{U*}"] \arrow[dr, "if\ \exists\phi_1"] & \\
\pi_1(U \cap V) \arrow[ur, "i_1"] \arrow[r, "inclusion"] \arrow[dr, "i_2"'] & \pi_1(X) \arrow[r, dashed, "then\ \exists!\Phi"] & H \\
 & \pi_1(V) \arrow[u, "i_{V*}"'] \arrow[ur, "if\ \exists\phi_2"'] &
\end{tikzcd}$$

Let $H = \{ i_u(c_1)^* i_v(c_1), \ldots \}$

N = least normal subj gn $\Rightarrow gNg^{-1} \in N$

Pf of Thm 59.1

If $g(I_k), g(I_{k+1}) \subset U$
$(I_k \cup I_{k+1})$ continue unti
$[a_\ell, a_{k+1}]$ $a_\ell \in U \Rightarrow a_{k+1} \in U$
or $a_\ell \in V \Rightarrow a_{k+1} \in V$
constant
$a_\ell \in U \cap V$



$g = \underbrace{\alpha_0 g_1 \alpha_1}_{\text{loop in } U} \underbrace{(\bar{\alpha}_1 g_2 \alpha_2)}_{\text{loop in } V} \underbrace{(\bar{\alpha}_2 g_3 \alpha_3)}_{\text{loop in } U}$

const

$\boxed{(\bar{\alpha}_i g_{i+1} \alpha_{i+1}) \text{ loop in } U \text{ or loop in } V}$

$g : S^1 \longrightarrow U \cup V = X$

$S^1 \subset g^{-1}(U) \cup g^{-1}(V)$

$\exists$ Lebesgue # δ st if $d(x_1, x_2) < \delta$ $x_1, x_2 \in S^1$

$\Rightarrow x_1, x_2 \in g^{-1}(U)$ or $x_1, x_2 \in g^{-1}(V)$

$(I_k) \in g^{-1}(U)$ or $\in g^{-1}(V)$

$g(I_k) \subset gg^{-1}(U) \subset U$ or $\subset gg^{-1}(V) \subset V$