If $H < G = \pi_1(X, x_0)$, $\exists p \colon \widetilde{X} \to X$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x_0})) = H$

Step 1: Define $\widetilde{X}$ by

1.) $P = \{[\alpha] \mid \alpha$ a **path** in X starting at $x_0\}$.

2.) $\widetilde{X} = P/\sim$ where $\alpha \sim \beta$ iff $\alpha\beta^{-1} \in H$.

Let $\alpha^{\#}$ denote the equivalence class of $\alpha \in P$.

Note: If $[\alpha] = [\beta]$, then $\alpha^{\#} = \beta^{\#}$ since $\alpha\beta^{-1} = e \in H$.

Note: If $\alpha^{\#} = \beta^{\#}$, then $(\alpha\delta)^{\#} = (\beta\delta)^{\#}$ when the product is defined since $(\alpha\delta)(\beta\delta)^{-1} = \alpha\delta\delta^{-1}\beta^{-1} = \alpha\beta^{-1} \in H$.

Define $p : \widetilde{X} \to X, \; p(\alpha^{\#}) = \alpha(1)$.

Note p is onto since X is path connected.

Step 2: Topologize E

Method 1: Give P the compact-open topology

$S(C, U) = \{\alpha \mid \alpha : [0,1] \to X$ such that $\alpha(C) \subset U\}$

$\mathcal{S} = \{S(C, U) \mid C$ compact in $[0, 1]$, U open in $X\}$ is subbases for compact-open topology on the set of paths P.

Give $\widetilde{X} = P/\sim$ the quotient topology.

Or equivalently,

Method 2: Let $\alpha \in P$, U path connected open nbhd of $\alpha(1)$.

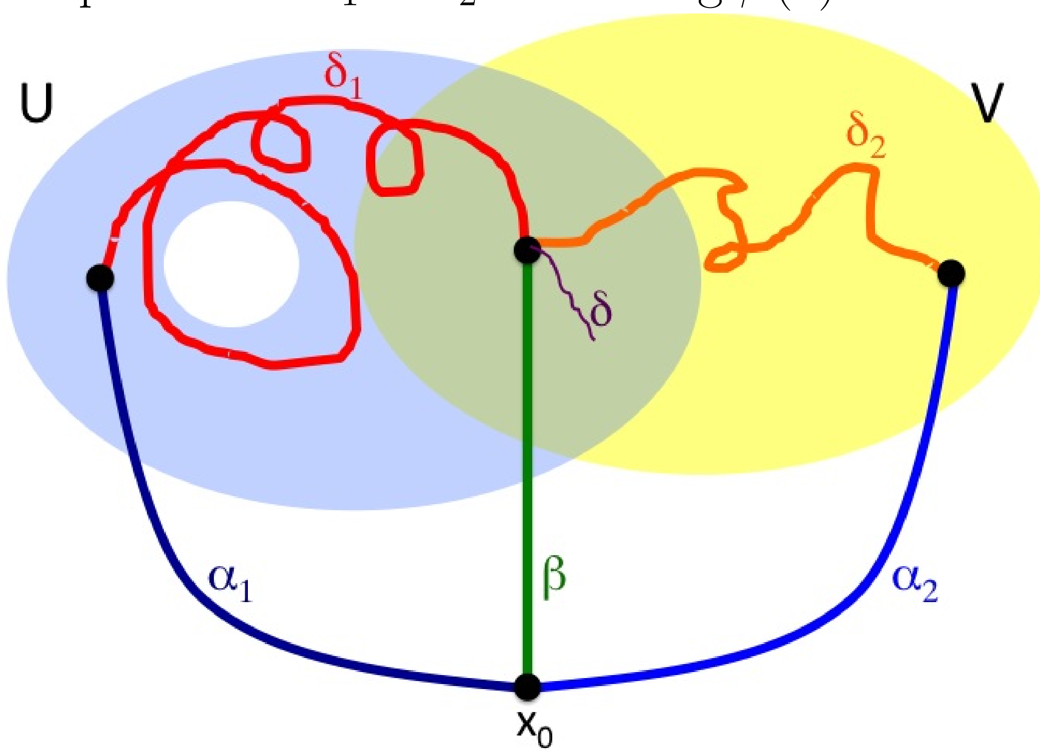
$$B(U, \alpha) = \{(\alpha * \delta)^\# \mid \delta \text{ path in } U \text{ such that } \delta(0) = \alpha(1)\}$$

Claim: $\{B(U, \alpha) \mid \alpha \in P, \ U$ path connected nbhd of $\alpha(1)\}$ is a basis for a topology on $\widetilde{X}$.

1.) $\alpha^\# = (\alpha * e_{\alpha(1))})^\# \in B(U, \alpha)$

2.) Suppose $\beta^\# \in B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$.

$\beta^\# \in B(V, \beta) \subset B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$ where V is path connected component of $U_1 \cap U_2$ containing $\beta(1)$.



$\beta^\# = (\alpha_1\delta_1)^\# = (\alpha_2\delta_2)^\#$ implies $(\beta\delta)^\# = (\alpha_1\delta_1\delta)^\# = (\alpha_2\delta_2\delta)^\#$

Claim: $\beta^\# \in B(U, \alpha)$ implies $B(U, \alpha) = B(U, \beta)$.

$\beta^\# \in B(U, \alpha)$ implies $\beta^\# = (\alpha\delta)^\#$ implies $\alpha^\# = (\beta\delta^{-1})^\#$. Thus $\alpha^\# \in B(U, \beta)$.

$(\beta\delta')^\# = (\alpha\delta\delta')^\# \in B(U, \alpha)$. Thus $B(U, \beta) \subset B(U, \alpha)$. Similarly, $B(U, \alpha) \subset B(U, \beta)$.

Thus $B(U, \alpha) \cap B(U, \beta) \neq \emptyset$ implies $B(U, \alpha) = B(U, \beta)$.

Step 3. p is open and continuous.

Note $p(B(U, \alpha)) = U$ since $p((\alpha\delta)^{\#}) = \delta(1)$ and U is path connected. Thus p is open.

Claim p continuous. Let W be open in X. Let $\alpha^{\#} \in p^{-1}(W)$. Let U be a path connected component of W such that $\alpha(1) \in U$. Note U is open. Then $\alpha^{\#} \in B(U, \alpha) \subset p^{-1}(W)$.

Step 4. Claim: $\forall z \in X$, $\exists U$ open in X such that U is evenly covered by p.

Let $z \in X$. Take U path connected nbhd of z such that

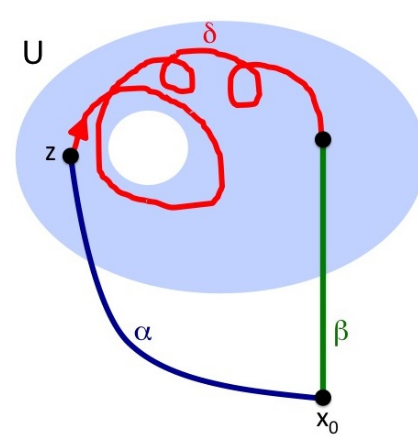
$$i_* : \pi_1(U, z) \to \pi_1(X, z) \text{ is trivial.}$$

Note U exists since X is path connected, locally path connected, and semilocally simply connected.

Claim: $p^{-1}(U) = \bigcup_{\alpha \text{ path from } x_0 \text{ to } z} B(U, \alpha)$

($\supset$): Clear since $p(B(U, \alpha)) = U$.

($\subset$): If $\beta^{\#} \in p^{-1}(U)$, then $\beta(1) \in U$.
Let δ be a path in U from z to $\beta(1)$.
Let $\alpha = \beta\delta^{-1}$ is a path from x_0 to z.
Thus $\beta^{\#} = (\alpha\delta)^{\#} \in B(U, \alpha)$.



Claim: $p|_{B(U,\alpha)} : B(U,\alpha) \to U$ is a bijection.

Onto: Recall $p(B(U,\alpha)) = U$.

1:1: Suppose $p((\alpha\delta_1)^{\#}) = p((\alpha\delta_2)^{\#})$ where $\delta_1, \delta_2 \subset U$.

Then $\delta_1(1) = \delta_2(1)$.

Recall $i_* : \pi_1(U, z) \to \pi_1(X, z)$ is trivial.

Thus $\delta_1\delta_2^{-1} = e$ in $\pi_1(X, z)$.

Thus $[\alpha\delta_1] = [\alpha\delta_2]$ and $(\alpha\delta_1)^{\#} = (\alpha\delta_2)^{\#}$.

Step 5, 6. $\widetilde{X}$ is path connected.

Let $\widetilde{x} = e_{x_0}$, the constant path at x_0. Then $p(\widetilde{x}) = e_{x_0}(1) = x_0$.

Let $\alpha^{\#} \in \widetilde{X}$.

Let $c \in [0, 1]$. Define $\alpha_c : I \to X$, $\alpha_c(t) = \alpha(tc)$.

Thus α_c is a path from x_0 to $\alpha(c)$.

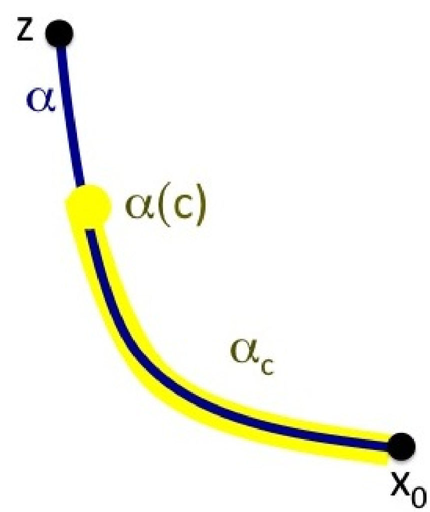
Define $\widetilde{\alpha} : I \to \widetilde{X}$ by $\widetilde{\alpha}(c) = (\alpha_c)^{\#}$.

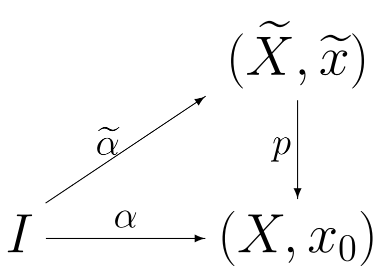
$p(\widetilde{\alpha}(c)) = p((\alpha_c)^{\#}) = \alpha_c(1) = \alpha(c)$.

Hence $p \circ \widetilde{\alpha} = \alpha$.

Thus $\widetilde{\alpha}$ is the lift of α starting at x_0

if $\widetilde{\alpha}$ is continuous.





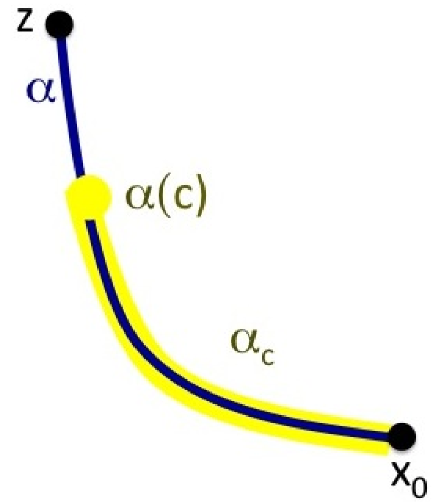
Claim: $\widetilde{\alpha} : I \to \widetilde{X}$ is continuous. Let $c \in I$.

$B(U, \alpha_c)$ is an arbitrary basis element containing $\widetilde{\alpha}(c) = \alpha_c^{\#}$.

$\alpha : I \to X$ is continuous.

Thus $\exists \varepsilon > 0$ such that if $t \in (c - \varepsilon, c + \varepsilon)$, then $\alpha(t) \in U$.

Then $\alpha_t^{\#} = (\alpha_c \delta)^{\#} \in B(U, \alpha_c)$



*Note compared to the standard $\delta - \varepsilon$ definition of continuity, our $\varepsilon =$ old δ and our U relates to the old ε.

Step 7. Claim $p_*(\pi_1(\widetilde{X}, \widetilde{x})) = H$.

$[\alpha] \in p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ iff $\widetilde{\alpha} \in \pi_1(\widetilde{X}, \widetilde{x})$

iff $\widetilde{\alpha}$ is a loop in $\widetilde{X}$ based at $\widetilde{x}$

iff $\alpha^{\#} = \widetilde{\alpha}(1) = \widetilde{x}$ iff $[\alpha] = [\alpha * e_{x_0}^{-1}] \in H$.

Thus $p_*(\pi_1(\widetilde{X}, \widetilde{x})) = H$