Proposition 1.

$$\begin{array}{ccc} \widetilde{X_1} & \overset{\text{then } \exists h}{- - \to} & \widetilde{X_2} \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & X & \end{array}$$

Suppose $p_1(\widetilde{x_1}) = p_2(\widetilde{x_2}) = x_0$ The covering maps p_1, and p_2 are equivalent iff the subgroups $(p_1)_*(\pi_1(\widetilde{X_1}, \widetilde{x_1}))$ and $(p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2}))$ are conjugate in $\pi_1(X, x_0)$.

Proof: $(\Rightarrow)$ Suppose h exists. Then by Prop 79.2,

$$(p_1)_*(\pi_1(\widetilde{X_1}, \widetilde{x_1})) = (p_2)_*(\pi_1(\widetilde{X_2}, h(\widetilde{x_1})))$$

$$\begin{array}{ccc} (\widetilde{X_1}, \widetilde{x_1}) & \overset{h}{\to} & (\widetilde{X_2}, h(\widetilde{x_1})) \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & (X, x_0) & \end{array}$$

By prop 79.3a, $(p_2)_*(\pi_1(\widetilde{X_2}, h(\widetilde{x_1})))$ and $(p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2}))$ are conjugate in $\pi_1(X, x_0)$.

Thus $(p_1)_*(\pi_1(\widetilde{X_1}, \widetilde{x_1}))$ and $(p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2}))$ are conjugate in $\pi_1(X, x_0)$.

$(\Leftarrow)$ Suppose $H_1 = (p_1)_*(\pi_1(\widetilde{X_1}, \widetilde{x_1}))$ and $H_2 = (p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2}))$ are conjugate in $\pi_1(X, x_0)$.

By 79.3b, there exists $\widetilde{x_2'} \in \widetilde{X_2}$ such that $H_1 = (p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2'}))$ Thus h exists by 79.2.

Definition 0.1. X is *semilocally simply connected* if $\forall x \in X, \exists U$ open in X such that $x \in U$ and the homomorphism induced by inclusion is trivial:

$$i_* : \pi_1(U, x) \to \pi_1(X, x), \qquad i([\alpha]) = [\alpha] = [e].$$

Proposition 2. If X has a universal cover, then X is semilocally simply-connected.

Proposition 3. Suppose X is a path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subset \pi_1(X, x_0)$ there is a covering space $p\colon \widetilde{X_H} \to X$ such that $p_*(\pi_1(\widetilde{X_H}, \widetilde{x_0})) = H$ for a suitably chosen basepoint $\widetilde{x_0} \in \widetilde{X_H}$.

Corollary 1. X has a universal cover iff X is path-connected, locally-path connected, and semilocally simply-connected.

$$F : \left\{ \begin{array}{c} \widetilde{X} \\ {\scriptstyle p}\big\downarrow \\ X \end{array} \;\middle|\; p \text{ is a covering map} \right\} \to \{H \mid H < G\}$$

$$F(p) = p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$$

Note since $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$ is a homomorphism, $p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$ is a subgroup of $\pi_1(X, x_0)$. Thus F is well-defined.