PROPOSITION 1.

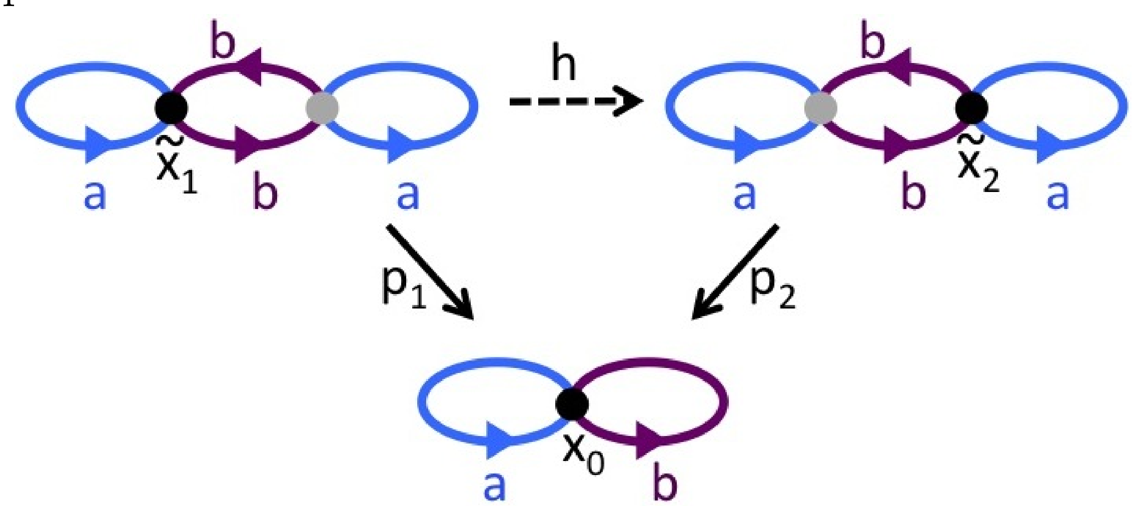
$$\begin{array}{ccc} (\widetilde{X_1}, \widetilde{x_1}) & \xrightarrow{\text{then } \exists ! h} & (\widetilde{X_2}, \widetilde{x_1}) \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & (X, x_0) & \end{array}$$

$p_1 \colon \widetilde{X_1} \to X$ and $p_2 \colon \widetilde{X_2} \to X$ are equivalent
via a homeomorphism $h \colon \widetilde{X_1} \to \widetilde{X_2}$
taking a basepoint $\widetilde{x_1} \in p_1^{-1}(x_0)$ to a basepoint $\widetilde{x_2} \in p_2^{-1}(x_0)$

if and only if

$$(p_1)_*(\pi_1(\widetilde{X_1}, \widetilde{x_1})) = (p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2})).$$

Example:

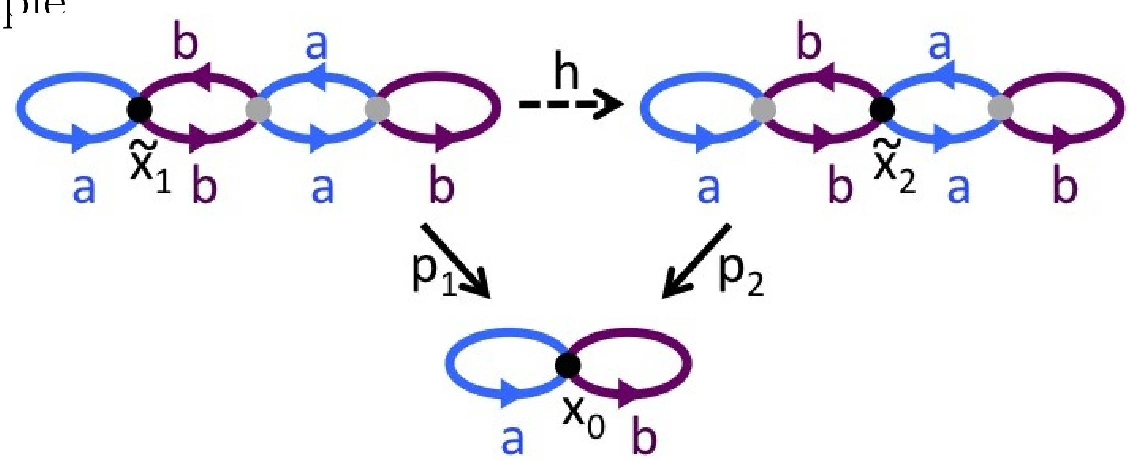


$$(p_1)_*(\pi_1(\widetilde{X_1}, \widetilde{x_1})) \;=\; < a, b, bab^{-1} > \;=\; (p_2)_*(\pi_1(\widetilde{X_2}, \widetilde{x_2})).$$

Proposition 2. Given covering map $\begin{array}{c}\widetilde{X}\\ p\big\downarrow\\ X\end{array}$ and $\widetilde{x_1}, \widetilde{x_2} \in p_1^{-1}(x_0)$, $p_*(\pi_1(\widetilde{X}, \widetilde{x_1}))$ and $p_*(\pi_1(\widetilde{X}, \widetilde{x_2}))$are conjugate in $\pi_1(X, x_0)$.

Moreover, let $H_1 = p_*(\pi_1(\widetilde{X}, \widetilde{x_1}))$ and $H_2 = p_*(\pi_1(\widetilde{X}, \widetilde{x_2}))$, let γ be a path in $\widetilde{X}$ from $\widetilde{x_1}$ to $\widetilde{x_2}$, and let $\alpha = p \circ \gamma \in \pi_1(X, x_0)$ then $H_0 = \alpha H_1 \alpha^{-1}$

Example:



$$H_1 = (p_1)_*(\pi_1(\widetilde{X}, \widetilde{x_1})) \;=\; < a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} >$$

$$H_2 = (p_2)_*(\pi_1(\widetilde{X}, \widetilde{x_2})) \;=\; < bab^{-1}, b^2, a^2, aba^{-1} >$$

$$bH_2b^{-1} \;=\; < bbab^{-1}b^{-1}, bb^2b^{-1}, ba^2b^{-1}, baba^{-1}b^{-1} >$$

$$= \; < b^2ab^{-2}, b^2, ba^2b^{-1}, baba^{-1}b^{-1} >$$

$a, b^2 \in H_1$ implies $b^2ab^{-2} \in H_1$

$b^2, b^2ab^{-2} \in bH_2b^{-1}$ implies $b^{-2}b^2ab^{-2}b^2 = a \in bH_2b^{-1}$

Thus $H_1 = bH_2b^{-1}$

Proposition 3. Given covering map $\begin{array}{c}(\widetilde{X},\widetilde{x_0})\\ \downarrow p\\ (X,x_0)\end{array}$, $H_0 = p_*(\pi_1(\widetilde{X},\widetilde{x_0}))$.

If H is a subgroup of $\pi_1(X,x_0)$, such that $H_0 = \alpha H \alpha^{-1}$, then

$\exists \widetilde{x_1} \in p_1^{-1}(x_0)$ such that $H = (p_1)_*(\pi_1(\widetilde{X_1},\widetilde{x_1}))$.

Proposition 4.

$$\begin{array}{ccc} \widetilde{X_1} & \overset{\text{then } \exists h}{\dashrightarrow} & \widetilde{X_2} \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

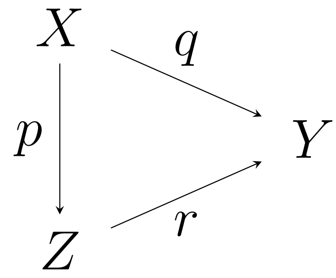
Suppose $p_1(\widetilde{x_1}) = p_2(\widetilde{x_2}) = x_0$ The covering maps p_1, and p_2 are equivalent iff the subgroups $(p_1)_*(\pi_1(\widetilde{X_1},\widetilde{x_1}))$ and $(p_2)_*(\pi_1(\widetilde{X_2},\widetilde{x_2}))$ are conjugate in $\pi_1(X,x_0)$.

Subgroups of $\mathbb{Z}$ are $\{e\}$, and $n\mathbb{Z}$, $n = 1, 2, 3, ...$

$$\begin{array}{c}\mathbb{R}\\ p\downarrow\\ S^1\end{array} \qquad \pi_1(\mathbb{R},0) = \{e\}. \text{ Thus } p_*(\pi_1(\mathbb{R},0)) = \{e\}$$

$$\begin{array}{c}S^1\\ p_n=z^n\downarrow\\ S^1\end{array} \qquad \pi_1(S^1,0) = \mathbb{Z} \text{ and } p_{n*}(\pi_1(S^1,0)) = n\mathbb{Z}$$

By the above proposition, any covering map of S^1 is equivalent to one of the covering maps above.



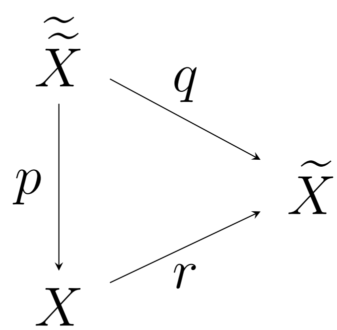
Proposition 5.

Let p, q, r be continuous maps with $p = r \circ q$. Then

i.) If q and r are covering maps and if either (Z has a universal covering space) or ($r^{-1}(z)$ is finite for all $z \in Z$), then p is a covering map.

ii.) If p and r are covering maps, then q is a covering map.

iii.) If p and q are covering maps, then r is a covering map.



Proposition 6. If p is a covering map and if $\pi_1(\widetilde{X}) = \{e\}$, then given any covering map r, there exists covering map q such that $p = r \circ q$.

Definition 0.1. If $\begin{array}{c}\widetilde{\widetilde{X}}\\ \downarrow p \\ X\end{array}$ is a covering map and if $\pi_1(\widetilde{\widetilde{X}}) = \{e\}$,

then $\widetilde{\widetilde{X}}$ is called the *universal covering space of* X.