Thm 3.26: Let M be a closed connected n-manifold. Then

(a) If M is R-orientable, the map

$$H_n(M;R) \to H_n(M, M - \{x\}; R) \cong R$$

is an isomorphism for all $x \in M$.

(b) If M is not R-orientable, the map

$$H_n(M;R) \to H_n(M, M - \{x\}; R) \cong R$$

is injective with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$.

(c) $H_n(M;R) = 0$.

Cor: $H_n(M^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & M^n \textit{ orientable} \\ 0 & \textit{else} \end{cases}$

Cor: $H_n(M^n, \mathbb{Z}_2) = \mathbb{Z}_2$

If $H_n(M;R) \cong R \cong < [M] >$, then $[M]$ is a fundamental class for M (with coefficients in R).

Lemma: A fundamental class exists iff M is closed and R-orientable.

Proof: ($\Rightarrow$) Suppose $H_n(M;R) \cong R \cong < [M] >$ where

$p_{x*} : H_n(M;R) \to H_n(M, M - \{x\}; R) \cong R$ is an isomorphism for all x.

Define a function $x \in M \to \mu_x = p_{x*}([M]) \in H_n(M, M - \{x\}; R)$

Sums are finite implies M compact.

($\Leftarrow$) Thm 3.26 implies that if M is R-orientable, then a fundamental class $[M]$ exists.

Poincaré Duality

Definition 1. *Let X be a space. The cap product is a pairing between certain homology groups and cohomology groups of X. For $k \geq \ell$, we define*

$$\frown : C_k(X; R) \times C^\ell(X; R) \to C_{k-\ell}(X; R)$$

$$\sigma \frown \phi = \phi(\sigma|_{[v_0 \cdots v_\ell]})\sigma|_{[v_\ell \cdots v_k]}.$$

Example (using simplicial instead of singular notation):

Suppose $\sigma = [v_0 \cdots v_k] \in C_k$ and $\phi : C_\ell \to \mathbb{Z}$ (i.e, $\phi \in C^\ell(X; \mathbb{Z})$)

If, for example, $\phi[v_0, ..., v_\ell] = 5$, then

$$\sigma \frown \phi = \phi([v_0 \cdots v_\ell])[v_\ell \cdots v_k] = 5[v_\ell \cdots v_k] \in C_{k-\ell}(X; \mathbb{Z}).$$

It is easy to check the following properties:

- $\frown$ is bilinear (by definition)
- $\partial(\sigma \frown \phi) = (-1)^\ell(\partial\sigma \frown \phi - \sigma \frown \delta\phi)$
- $\frown(Z_k \times Z^\ell) \subseteq Z_{k-\ell}$, i.e. cycle $\frown$ cocycle = cycle
- $\frown(B_k \times Z^\ell) \subseteq B_{k-\ell}$, i.e. boundary $\frown$ cocycle = boundary

  $\delta(\phi) = 0$ implies $\partial(\sigma \frown \phi) = (-1)^\ell(\partial\sigma \frown \phi)$
- $\frown(Z_k \times B^\ell) \subseteq Z_{k-\ell}$, i.e. cycle $\frown$ coboundary = boundary

  $\partial(\sigma) = 0$ implies $\partial(\sigma \frown \phi) = (-1)^{\ell+1}(\sigma \frown \delta\phi)$

These facts imply that the cap product descends to a bilinear map

$$\frown : H_k(X) \times H^\ell(X) \to H_{k-\ell}(X).$$

$$\partial(\sigma \frown \phi) = \partial(\phi([v_0 \cdots v_\ell])[v_\ell \cdots v_k]) = \phi([v_0 \cdots v_\ell])\partial([v_\ell \cdots v_k])$$
$$= \phi([v_0, ..., v_\ell) \sum_{i=\ell}^{k} (-1)^{i-\ell}[v_\ell, ..., \widehat{v_i}, ..., v_k]$$
$$= \sum_{i=\ell}^{k} (-1)^{i-\ell} \phi([v_0, ..., v_\ell)[v_\ell, ..., \widehat{v_i}, ..., v_k]$$

$$\partial\sigma \frown \phi = \sum_{i=0}^{k} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_k] \frown \phi =$$
$$= \sum_{i=0}^{\ell} (-1)^i \phi([v_0, ..., \widehat{v_i}, ..., v_{\ell+1}])[v_{\ell+1}, ..., v_k]$$
$$+ \sum_{i=\ell+1}^{k} (-1)^i \phi([v_0, ..., v_\ell])[v_\ell, ..., \widehat{v_i}, ..., v_k]$$

$$\sigma \cap \delta\phi = \delta\phi([v_0 \cdots v_{\ell+1}])[v_{\ell+1} \cdots v_k] = \phi\partial([v_0 \cdots v_{\ell+1}])[v_{\ell+1} \cdots v_k]$$
$$= \phi(\sum_{i=0}^{\ell+1} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_{\ell+1}])[v_{\ell+1} \cdots v_k]$$
$$= \sum_{i=0}^{\ell+1} (-1)^i \phi([v_0, ..., \widehat{v_i}, ..., v_{\ell+1}])[v_{\ell+1} \cdots v_k]$$

Theorem 1 (Poincaré Duality)**.** *Let M be a closed, R-orientable n-manifold with fundamental class $[M] \in H_n(M; R)$. Then the map $D_M : H^i(M) \to H_{n-i}(M)$ is an isomorphism, where D_M is defined by*
$$D_M([\phi]) = [M] \frown [\phi].$$

Corollary 1. *Let M be a closed, connected, R-orientable n-manifold. The top homology group $H_n(M)$ is isomorphic to $\mathbb{Z}$, and $[M]$ is a generator.*