Recall $\prod_n H^n(X;G)$ is an abelian group under addition.

$([\phi_0],[\phi_1],[\phi_2],...) \in \prod_n H^n(X;G)$

where $\phi_n : C_n \to G \ \in \ ker(\delta) \ = \ Z^*_n(X;G) \ \subset \ C^n(X;G)$

$\phi \sim \psi \in H^n(X;G)$ iff $\phi = \psi + \delta\chi$ for some $\chi \in C^{n+1}(X;G)$.

Defn: A is a **graded ring** if $A = \bigoplus_{k\geq 0} A_k$ where

A_k are additive groups with multiplication $m : A_k \times A_\ell \to A_{k+\ell}$

Defn: $a_k \in A_k$ has **grade = degree = dimension** $= |a_k| = k$

Examples: Polynomial rings.

$R[x] = \{r_0 + r_1x + ... + r_kx^k \mid r_i \in R, k \in \mathbb{N}\}$

$R[x]/(x^n) = \{r_0 + r_1x + ... + r_kx^k \mid r_i \in R, k \in \{0,...,n-1\}\}$

where $x^ix^j = x^m$ where $m = i + j \ mod \ n$.

$R[x_1,...,x_n]$ where $rx_1^{i_1}x_2^{i_2}...x_\ell^{i_n}$ has grade $\sum_{j=1}^{n} i_j$

Example: Exterior Algebras $\Lambda_R[x_1,...,x_n]$

Basis for grade k: $x_{i_1}x_{i_2}...x_{i_k}$ where $i_1 < i_2 < ... < i_k$

and $x_ix_j = -x_jx_i$ and $x_i^2 = 1$ for all i, j

Cup Product Let R be a commutative ring with identity.

Defn: For cochains $\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$,
the **cup product** $\phi \smile \psi \in C^{k+\ell}(X;R)$ is the cochain defined by

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,v_{k+\ell}]}).$$

where $\sigma \in C_{k+\ell}(X)$.

The cup product is associative:

The cup product is distributive:

Thus if R is a ring, $\bigoplus_n C^n(X;R)$ is a ring.

If R has a multiplicative identity, 1, then $\iota : C_0(X) \to R$, $\iota(v) = 1$ for all vertices v is the multiplicative identity for $\bigoplus_n C^n(X;G)$:

$$(\iota \smile \psi)(\sigma) = \iota(\sigma|_{[v_0]}) \cdot \psi(\sigma|_{[v_0,\dots,v_\ell]}) = 1 \cdot \psi(\sigma|_{[v_0,\dots,v_\ell]}) = \psi(\sigma)$$

$$(\phi \smile \iota)(\sigma) = \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \iota(\sigma|_{[v_k]}) = \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot 1 = \phi(\sigma).$$

Proposition 1. *For $\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, we have*

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi.$$

Note: The cup product of two cocycles is a cocycle.

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi = 0 + 0 = 0.$$

The cup product of a cocycle and a coboundary is a coboundary.

$$(-1)^k \phi \smile \delta\psi = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi = \delta(\phi \smile \psi).$$

The cup product of a coboundary and a cocycle is a coboundary.

$$\delta\phi \smile \psi = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi = \delta(\phi \smile \psi).$$

Theorem 1. *When R is commutative, the rings $H^\bullet(X, A; R)$ are graded commutative, that is, the identity*

$$\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$$

holds for all $\alpha \in H^k(X, A; R)$ and $\beta \in H^\ell(X, A; R)$.

$$H^n(S^2 \vee S^4; \mathbb{Z}) \cong H^n(\mathbb{CP}^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0, 2, 4 \\ 0 & else \end{cases}$$

But $\cup : H^2(S^2 \vee S^4; \mathbb{Z}) \times H^2(S^2 \vee S^4; \mathbb{Z}) \to H^4(S^2 \vee S^4; \mathbb{Z})$ is the zero map,

while for $\cup : H^2(\mathbb{CP}^2; \mathbb{Z}) \times H^2(\mathbb{CP}^2; \mathbb{Z}) \to H^4(\mathbb{CP}^2; \mathbb{Z}),\ \cup(1, 1) = 1.$

I.e, the group structure does not distinguish $S^2 \vee S^4$ from $\mathbb{CP}^2$, but the ring structure does.

Proposition 2. *For a map $f : X \to Y$, the induced maps $f^* : H^n(Y; R) \to H^n(X; R)$ satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$, and similarly in the relative case.*

Thus $H^\bullet(-; R)$ is a functor from the category of topological spaces to the category of graded R-algebras.

$$H^\bullet(\mathbb{RP}^n;\mathbb{Z}/2) \cong (\mathbb{Z}/2)[\alpha]/(\alpha^{n+1}), \quad |\alpha| = 1 \tag{1}$$
$$H^\bullet(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong (\mathbb{Z}/2)[\alpha], \quad |\alpha| = 1 \tag{2}$$
$$H^\bullet(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}), \quad |\alpha| = 2 \tag{3}$$
$$H^\bullet(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha], \quad |\alpha| = 2 \tag{4}$$
$$H^\bullet(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha], \quad |\alpha| = 2 \tag{5}$$
$$H^\bullet(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}), \quad |\alpha| = 4H^\bullet(\mathbb{HP}^\infty;\mathbb{Z}) \tag{6}$$

The cohomology of real projective spaces over $\mathbb{Z}$ are slightly more delicate.

$$H^\bullet(\mathbb{RP}^{2n};\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{n+1}), \quad |\alpha| = 2 \tag{7}$$
$$H^\bullet(\mathbb{RP}^{2n+1};\mathbb{Z}) \cong \mathbb{Z}[\alpha,\beta]/(2\alpha, \alpha^{n+1}, \beta^2, \alpha\beta), \quad |\alpha| = 2, |\beta| = 2n+1 \tag{8}$$
$$H^\bullet(\mathbb{RP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha), \quad |\alpha| = 2 \tag{9}$$

Proposition 3. *The isomorphisms*

$$H^\bullet(\textstyle\coprod_\alpha X_\alpha; R) \longrightarrow \prod_\alpha H^\bullet(X_\alpha; R)$$
$$\widetilde{H}^\bullet(\textstyle\bigvee_\alpha X_\alpha; R) \longrightarrow \prod_\alpha \widetilde{H}^\bullet(X_\alpha; R)$$

whose coordinates are induced by the inclusions $i_\alpha\colon X_\alpha \hookrightarrow \coprod_\alpha X_\alpha$ *and* $i_\alpha\colon X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ *respectively are ring isomorphisms.*

The second isomorphism above provides us with a tool to reject spaces as being homotopy equivalent to wedge products.

Defn: Let $(A, +)$ and $(B, +)$ be abelian groups. Then the tensor product is the abelian group $(A \otimes B, +)$ such that

1. There is a bilinear map $i \colon A \times B \to A \otimes B$ and
2. Given any bilinear map $f \colon A \times B \to C$, there is a unique linear map $L_f \colon A \otimes B \to C$ such that $L_f \circ i = f$.

Defn: $A \otimes B =< a \otimes b \mid (a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b,$
$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2 >$

Defn: Let R be a commutative ring and $(A, +)$ and $(B, +)$ be R-modules. Then the tensor product is the R-module $(A \otimes_R B, +)$ such that

1. There is a bilinear map $i \colon A \times B \to A \otimes_R B$ and
2. Given any bilinear map $f \colon A \times B \to C$, there is a unique linear map $L_f \colon A \otimes_R B \to C$ such that $L_f \circ i = f$.

Defn: $A \otimes_R B =< a \otimes_R b \mid (a_1 + a_2) \otimes_R b = a_1 \otimes_R b + a_2 \otimes_R b,$
$a \otimes_R (b_1 + b_2) = a \otimes_R b_1 + a \otimes_R b_2, ra \otimes_R b = a \otimes_R rb >$

Ex: If R has an identity 1, $R \otimes_R A \cong R$.

$\phi(r_1 \otimes a) = r_1 a$. with inverse $\psi : R \to R \otimes_R A$, $\psi(a) = 1 \otimes a$.

$\phi(\psi(a)) = \phi(1 \otimes a) = a.$

$\psi(\phi(r \otimes a) = \psi(ra) = 1 \otimes ra = r \otimes a.$

Useful facts:

1.) $A \otimes B \cong B \otimes A$.

2.) $(\bigoplus_i A_i) \otimes B \cong \bigoplus_i (A_i \otimes B)$.

3.) $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

4.) $\mathbb{Z} \otimes A \cong A$ via isomorphism $\phi(n \otimes a) = na$.

with inverse $\psi : A \to \mathbb{Z} \otimes A$, $\psi(a) = 1 \otimes a$.

$$\phi(\psi(a)) = \phi(1 \otimes a) = a.$$

$$\psi(\phi(n \otimes a) = \psi(na) = 1 \otimes na = n \otimes a.$$

5.) $\mathbb{Z}_n \otimes A \cong A/nA$ via the isomorphism $\phi(n \otimes a) = na$.

Ex: $\mathbb{Z}_n \otimes \mathbb{Z} \cong Z_n$ while $\mathbb{Z}_n \otimes \mathbb{Q} \cong 0$

6.) Homomorphisms $f_i : A_i \to B_i$ induce a homomorphism $f_1 \otimes f_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$, $(f_1 \otimes f_2)(a_1 \otimes a_2) = f_1(a_1) \otimes f_2(a_2)$.

7.) A bilinear map $\phi : A \times B \to C$ induces a homomorphism $A \otimes B \to C$, sending $a \otimes b$ to $\phi(a, b)$.

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi} & C \\ {\scriptstyle \otimes}\downarrow & \nearrow_{\exists ! \phi} & \\ A \otimes B & & \end{array}$$

8.) $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact implies

$$A \otimes G \xrightarrow{f \otimes i_G} B \otimes G \xrightarrow{g \otimes i_G} C \otimes G \to 0 \text{ exact}$$

Let R be a commutative ring. In cohomology, we can create a product via the cross product . Recall induced maps:

$p_1 : X \times Y \to X. \qquad p_2 : X \times Y \to Y$

$p_1^* : H^k(X;R) \to H^k(X \times Y;R),$

If $a \in H^k(X;R)$, then $a : C_k(X) \to R$

Thus $p_1^*(a) = a \circ p_1 : C_k(X \times Y) \to R$

$$\Delta^k \xrightarrow{\sigma} X \times Y \xrightarrow{p_1} X \xrightarrow{a} R$$

$$(\Delta^k \xrightarrow{p_1 \circ \sigma} X) \xrightarrow{a} R \text{ implies } (\Delta^k \xrightarrow{\sigma} X \times Y) \xrightarrow{a \circ p_1} R$$

That is $p_1^*(k-\text{cocycle in } X) = k-\text{cocycle in } X \times Y$.

Thus we can define the cross product = external cup product:

$$H^k(X) \times H^\ell(Y) \to H^{k+\ell}(X \times Y).$$

$$a \times b \to p_1^*(a) \smile p_2^*(b)$$

$$(p_1^*(a) \smile p_2^*(b))(\sigma) = p_1^*(a)(\sigma|_{[v_0,\ldots,v_k]}) \cdot p_2^*(b)(\sigma|_{[v_k,\ldots,v_{k+\ell}]}).$$

for $\sigma \in C_n(X \times Y), \quad a : C_k(X) \to R, \quad b : C_\ell(Y) \to R$

$p_1^*(a) : C_k(X \times Y) \to R, \quad p_2^*(b) : C_\ell(X \times Y) \to R$

$p_1^*(a) \smile p_2^*(b) : C_{k+\ell}(X \times Y) \to R$

$$(\Delta^{k+\ell} \xrightarrow{\sigma} X \times Y) \longrightarrow p_1^*(a)(\sigma|_{[v_0,\ldots,v_k]}) \cdot p_2^*(b)(\sigma|_{[v_k,\ldots,v_{k+\ell}]}) \in R.$$

$\cup$ distributive implies $\times$ is bilinear.

Note if $s \in R$,

$$s(p_1^*(a) \smile p_2^*(b)) = (sp_1^*(a)) \smile p_2^*(b) = p_1^*(a) \smile s(p_2^*(b))$$

We have a ring homomorphism: $H^k(X) \otimes_R H^\ell(Y) \to H^{k+\ell}(X \times Y)$.

$$\begin{array}{ccc} H^*(X;R) \times H^*(Y;R) & \xrightarrow{\times} & H^*(X \times Y;R) \\ \Big\downarrow \otimes & \nearrow \exists! \phi & \\ H^*(X;R) \otimes H^*(X;R) & & \end{array}$$

Let $\mu(a \otimes b) = a \times b$

Defn: $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$ where $|x|$ = dimension of x.

$$\mu((a \otimes b)(c \otimes d)) = (-1)^{|b||c|}\mu(ac \otimes bd) = (-1)^{|b||c|}(ac) \times (bd)$$

$$= (-1)^{|b||c|} p_1^*(ac) \smile p_2^*(bd)$$

$$= (-1)^{|b||c|} p_1^*(a \smile c) \smile p_2^*(b \smile d)$$

$$= (-1)^{|b||c|} p_1^*(a) \smile p_1^*(c) \smile p_2^*(b) \smile p_2^*(d)$$

$$= (-1)^{|b||c|}(-1)^{|b||c|} p_1^*(a) \smile p_2^*(b) \smile p_1^*(c) \smile p_2^*(d)$$

$$= p_1^*(a) \smile p_2^*(b) \smile p_1^*(c) \smile p_2^*(d)$$

$$= (a \times b) \smile (c \times d) = \mu(a \otimes b) \smile \mu(c \otimes d)$$

$$=$$

Theorem 2 (Künneth formula)**.** *The cross product* $H^\bullet(X;R) \otimes_R H^\bullet(Y;R) \to H^\bullet(X \times Y;R)$ *is an isomorphism of rings if* X *and* Y *are CW complexes and* $H^k(Y;R)$ *is a finitely generated free* R*-module for all* k*.*

The hypothesis X and Y are CW complexes is unnecessary.

The result also holds in a relative setting.

Example: $H^\bullet(X, A; R) \otimes_R H^\bullet(Y;R) \to H^\bullet(X \times Y, A \times Y; R)$

$\Delta^k \xrightarrow{\sigma} A \times Y \xrightarrow{p_1} A \to R$

Theorem 3 (Relative Künneth formula)**.** *For CW pairs* (X, A) *and* (Y, B) *the cross product homomorphism* $H^\bullet(X, A; R) \otimes_R H^\bullet(Y, B; R) \to H^\bullet(X \times Y, A \times Y \cup X \times B; R)$ *is an isomorphism of rings if* $H^k(Y, B; R)$ *is a finitely generated free* R*-module for each* k*.*

Example: The continuous map $f : X \to X \times X$, $f(x) = (x, x)$ induces a homomorphism $f^* : H^n(X \times X; R) \to H^n(X; R)$ by if $\phi : C_n(X \times X) \to R$, then $\phi \circ f : C_n(X) \to R$.

$$\begin{array}{ccccc} H^*(X;R) \times H^*(X;R) & \xrightarrow{\times} & H^*(X \times X; R) & \xrightarrow{f^*} & H^*(X;R) \\ \downarrow \otimes & \nearrow \exists! \phi & & & \\ H^*(X;R) \otimes H^*(X;R) & & & & \end{array}$$

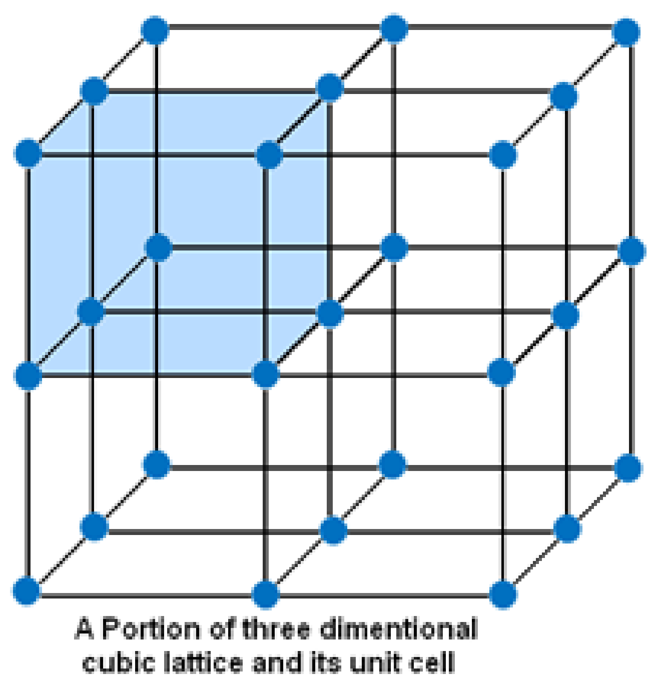
$\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} X \times X \xrightarrow{p_i} X \xrightarrow{a} R$

$p_i \circ f = id$, thus $f^* \circ p_i^* = id$

$$\text{Thus } H^k(X) \times H^\ell(X) \to H^{k+\ell}(X \times X) \to H^{k+\ell}(X).$$

$$a \times b \to a \smile b$$

Poincare duality



https://www.excellup.com/class_12/chemistry_12/SolidState_CrystalLattice.aspx

Poincaré Duality

Definition 10. *Let X be a space. The cap product is a pairing between certain homology groups and cohomology groups of X. For $k \geq \ell$, we define*

$$\frown : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$$
$$\sigma \frown \phi = \phi(\sigma|_{[v_0 \cdots v_\ell]})\sigma|_{[v_\ell \cdots v_k]}.$$

Example (using simplicial instead of singular notation):

Suppose $\sigma = [v_0 \cdots v_k] \in C_k$ and $\phi : C_\ell \to \mathbb{Z}$ (i.e, $\phi \in C^\ell(X;\mathbb{Z})$)

If, for example, $\phi[v_0, ..., v_\ell] = 5$, then

$$\sigma \frown \phi = \phi([v_0 \cdots v_\ell])[v_\ell \cdots v_k] = 5[v_\ell \cdots v_k] \in C_{k-\ell}(X;\mathbb{Z}).$$

It is easy to check the following properties:

- $\frown$ is bilinear (by definition)
- $\partial(\sigma \frown \phi) = (-1)^\ell(\partial\sigma \frown \phi - \sigma \frown \delta\phi)$
- $\frown(Z_k \times Z^\ell) \subseteq Z_{k-\ell}$, i.e. cycle $\frown$ cocycle = cycle
- $\frown(B_k \times Z^\ell) \subseteq B_{k-\ell}$, i.e. boundary $\frown$ cocycle = boundary
$$\delta(\phi) = 0 \text{ implies } \partial(\sigma \frown \phi) = (-1)^\ell(\partial\sigma \frown \phi)$$
- $\frown(Z_k \times B^\ell) \subseteq Z_{k-\ell}$, i.e. cycle $\frown$ coboundary = boundary
$$\partial(\sigma) = 0 \text{ implies } \partial(\sigma \frown \phi) = (-1)^{\ell+1}(\sigma \frown \delta\phi)$$

These facts imply that the cap product descends to a bilinear map

$$\frown : H_k(X) \times H^\ell(X) \to H_{k-\ell}(X).$$

$$\partial(\sigma \frown \phi) = \partial(\phi([v_0 \cdots v_\ell])[v_\ell \cdots v_k]) = \phi([v_0 \cdots v_\ell])\partial([v_\ell \cdots v_k])$$
$$= \phi([v_0, ..., v_\ell) \sum_{i=\ell}^{k} (-1)^{i-\ell}[v_\ell, ..., \widehat{v_i}, ..., v_k]$$
$$= \sum_{i=\ell}^{k} (-1)^{i-\ell}\phi([v_0, ..., v_\ell)[v_\ell, ..., \widehat{v_i}, ..., v_k]$$

$$\partial\sigma \frown \phi = \sum_{i=0}^{k} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_k] \frown \phi =$$
$$= \sum_{i=0}^{\ell} (-1)^i \phi([v_0, ..., \widehat{v_i}, ..., v_{\ell+1}])[v_{\ell+1}, ..., v_k]$$
$$+ \sum_{i=\ell+1}^{k} (-1)^i \phi([v_0, ..., v_\ell])[v_\ell, ..., \widehat{v_i}, ..., v_k]$$

$$\sigma \cap \delta\phi = \delta\phi([v_0 \cdots v_{\ell+1}])[v_{\ell+1} \cdots v_k] = \phi\partial([v_0 \cdots v_{\ell+1}])[v_{\ell+1} \cdots v_k]$$
$$= \phi(\sum_{i=0}^{\ell+1} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_{\ell+1}])[v_{\ell+1} \cdots v_k]$$
$$= \sum_{i=0}^{\ell+1} (-1)^i \phi([v_0, ..., \widehat{v_i}, ..., v_{\ell+1}])[v_{\ell+1} \cdots v_k]$$

Theorem 4 (Poincaré Duality)**.** *Let M be a closed, R-orientable n-manifold with fundamental class $[M] \in H_n(M; R)$. Then the map $D_M : H^i(M) \to H_{n-i}(M)$ is an isomorphism, where D_M is defined by*

$$D_M([\phi]) = [M] \frown [\phi].$$

Corollary 1. *Let M be a closed, connected, R-orientable n-manifold. The top homology group $H_n(M)$ is isomorphic to $\mathbb{Z}$, and $[M]$ is a generator.*