

Creating a cell complex = CW complex

Building block: $n\text{-cells} = \{x \text{ in } \mathbb{R}^n : ||x|| \leq 1\}$

Examples: 0-cell = $\{x \text{ in } \mathbb{R}^0 : ||x|| < 1\}$ •

1-cell = open interval = $\{x \text{ in } \mathbb{R} : ||x|| < 1\}$ (—)

2-cell = open disk = $\{x \text{ in } \mathbb{R}^2 : ||x|| < 1\}$ ●

3-cell = open ball = $\{x \text{ in } \mathbb{R}^3 : ||x|| < 1\}$ ●

Cell complex = CW complex

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Grading = dimension

$\partial_n(n\text{-cells}) = \{x \text{ in } \mathbb{R}^n : ||x|| = 1\}$

Example: disk = $\{x \text{ in } \mathbb{R}^2 : ||x|| \leq 1\}$

Simplicial complex
3 vertices, 3 edges, 1 triangle ● = ●●●

Cell complex
1 vertex, 1 edge, 1 disk. ● = ● U (—) U ●

● → ● U (—)

||

● → ● U ● = ●

Cell complex = CW complex

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X^0 = set of points with discrete topology.

Given the $(n-1)$ -skeleton X^{n-1} , form the n -skeleton, X_n , by attaching n -cells via maps $\sigma_\alpha: \partial D^n \rightarrow X^{n-1}$,
i.e., $X^n = X^{n-1} \cup D_\alpha^n / \sim$ where $x \sim \sigma_\alpha(x)$ for all $x \text{ in } \partial D_\alpha^n$

Example: disk = $\{x \text{ in } \mathbb{R}^2 : ||x|| \leq 1\}$

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3 vertices, 3 edges, 1 triangle ● = ●●●

Cell complex
1 vertex, 1 edge, 1 disk. ● = ● U (—) U ●

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||

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Example: sphere = $\{x \text{ in } \mathbb{R}^3 : ||x|| = 1\}$

Simplicial complex ● = ●●●●

Cell complex ● = ● U ●

● → ● U ● = ●

Fist image from <http://openclipart.org/detail/1000/a-raised-fist-by-liftern>

Example: constant, identity, constant maps

Cell complex
1 vertex, 1 edge, 2 disks. ● U (—) U ● U ●

● → ● U (—)

||

● → ● U ● = ●

→ ● U ● = ●

Let X be a CW complex.

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Given the $(n-1)$ -skeleton X^{n-1} , form the n -skeleton, X_n , by attaching n -cells via attaching maps $\sigma_\alpha: \partial D_\alpha^n \rightarrow X^{n-1}$,
i.e., $X^n = X^{n-1} \cup D_\alpha^n / \sim$ where $x \sim \sigma_\alpha(x)$ for all $x \text{ in } \partial D_\alpha^n$

The **characteristic map** $\Phi_\alpha: D_\alpha^n \rightarrow X$ is the map that extends the attaching map $\sigma_\alpha: \partial D_\alpha^n \rightarrow X^{n-1}$ and $\Phi_\alpha|_{D_\alpha^n}$ onto its image is a homeomorphism.
 Φ_α is the composition $D_\alpha^n \rightarrow X^{n-1} \cup D_\beta^n \rightarrow X^n \rightarrow X$

Your name homology

3 ingredients:

- 1.) Objects
- 2.) Grading
- 3.) Boundary map

Grading

Grading: Each object is assigned a unique grade.

Let $X_n = \{x_1, \dots, x_k\}$ = generators of grade n .

Extend grading on the set of generators to the set of n -chains: C_n = set of n -chains = $R[X_n]$

Normally n -chains in C_n are assigned to the grade n .

$\partial_n : C_n \rightarrow C_{n-1}$ such that $\partial^2 = 0$

$$C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \rightarrow \dots \rightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

$$H_n = Z_n/B_n = (\text{kernel of } \partial_n) / (\text{image of } \partial_{n+1})$$

$$= \frac{\text{null space of } M_n}{\text{column space of } M_{n+1}}$$

Rank $H_n = \text{Rank } Z_n - \text{Rank } B_n$

Čech homology

Given $\bigcup_{\alpha \in A} V_\alpha$ where V_α open for all α in A .

Objects = finite intersections = $\{\bigcap_{i=1}^n V_{\alpha_i} : \alpha_i \in A\}$

Grading = n = depth of intersection.

$$\partial_{n+1} \left(\bigcap_{i=1}^n V_{\alpha_i} \right) = \sum_{j=1}^n \left(\bigcap_{i \neq j}^n V_{\alpha_i} \right)$$

Ex: $\partial_0(V_\alpha) = 0$, $\partial_1(V_\alpha \cap V_\beta) = V_\alpha + V_\beta$

$$\partial_2(V_\alpha \cap V_\beta \cap V_\gamma) = (V_\alpha \cap V_\beta) + (V_\alpha \cap V_\gamma) + (V_\beta \cap V_\gamma)$$

Creating the Čech simplicial complex

1.) $B_1 \cap \dots \cap B_{k+1} \neq \emptyset$, create k -simplex $\{v_1, \dots, v_{k+1}\}$.

Unoriented simplicial complex using \mathbb{Z}_2 coefficients

0-simplex = vertex = v Grading = dimension

1-simplex = edge = $\{v_1, v_2\}$

$v_1 \text{ --- } e \text{ --- } v_2$ Note that the boundary of this edge is $v_2 + v_1$

2-simplex = face = $\{v_1, v_2, v_3\}$

$v_1 \text{ --- } e_1 \text{ --- } v_2$
 $v_2 \text{ --- } e_2 \text{ --- } v_3$
 $v_3 \text{ --- } e_3 \text{ --- } v_1$ Note that the boundary of this face is the cycle $e_1 + e_2 + e_3$

$$= \{v_1, v_2\} + \{v_2, v_3\} + \{v_1, v_3\}$$

Nerve Lemma: If V is a finite collection of subsets of X with all non-empty intersections of subcollections of V contractible, then $N(V)$ is homotopic to the union of elements of V .

<http://www.math.upenn.edu/~ghrist/EAT/EATchapter2.pdf>

Theorem: The choice of triangulation does not affect the homology.

$\partial_n : C_n \rightarrow C_{n-1}$ such that $\partial^2 = 0$

$$C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \rightarrow \dots \rightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

$$H_n = Z_n/B_n = (\text{kernel of } \partial_n) / (\text{image of } \partial_{n+1})$$

$$= \frac{\text{null space of } M_n}{\text{column space of } M_{n+1}}$$

Rank $H_n = \text{Rank } Z_n - \text{Rank } B_n$

Building blocks for oriented simplicial complex

3-simplex =

$$\sigma = (v_1, v_2, v_3, v_4) = (v_2, v_3, v_1, v_4) = (v_3, v_1, v_2, v_4)$$

$$= (v_2, v_1, v_4, v_3) = (v_3, v_2, v_4, v_1) = (v_1, v_3, v_4, v_2)$$

$$= (v_4, v_2, v_1, v_3) = (v_4, v_3, v_2, v_1) = (v_4, v_1, v_3, v_2)$$

$$= (v_1, v_4, v_2, v_3) = (v_2, v_4, v_3, v_1) = (v_3, v_4, v_1, v_2)$$

$-\sigma =$

$$(v_2, v_1, v_3, v_4) = (v_3, v_2, v_1, v_4) = (v_1, v_3, v_2, v_4)$$

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Čech homology

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$$\partial_{n+1} \left(\bigcap_{i=1}^n V_{\alpha_i} \right) = \sum_{j=1}^n \left(\bigcap_{i \neq j}^n V_{\alpha_i} \right)$$

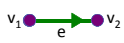
Ex: $\partial_0(V_\alpha) = 0$, $\partial_1(V_\alpha \cap V_\beta) = V_\alpha + V_\beta$

$$\partial_2(V_\alpha \cap V_\beta \cap V_\gamma) = (V_\alpha \cap V_\beta) + (V_\alpha \cap V_\gamma) + (V_\beta \cap V_\gamma)$$

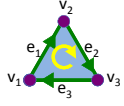
Oriented simplicial complex

0-simplex = vertex = v Grading = dimension

1-simplex = oriented edge = (v₁, v₂)
 Note that the boundary of this edge is v₂ - v₁



2-simplex = oriented face = (v₁, v₂, v₃)
 Note that the boundary of this face is the cycle
 e₁ + e₂ + e₃



$(v_1, v_2) + (v_2, v_3) - (v_1, v_3)$
 $= (v_1, v_2) - (v_1, v_3) + (v_2, v_3)$

$$\partial_n(v_0, \dots, v_n) = \sum_{i=0}^n (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_n) \text{ where } \hat{v}_i \text{ means omit } v_i$$

$$\partial_{n-1}(\partial_n(v_0, \dots, v_n)) = \partial_{n-1}(\sum_{i=0}^n (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_n))$$

$$= \sum_{i=0}^n (-1)^i \partial_{n-1}(v_0, \dots, \hat{v}_i, \dots, v_n)$$

$$= \sum_{i=0}^n (-1)^i \left[\sum_{k=0}^{i-1} (-1)^k (v_0, \dots, \hat{v}_k, \dots, \hat{v}_i, \dots, v_n) + \sum_{k=i+1}^n (-1)^k (v_0, \dots, \hat{v}_i, \dots, \hat{v}_k, \dots, v_n) \right]$$


$$= \sum_{k=0}^{i-1} (-1)^{i+k} (v_0, \dots, \hat{v}_k, \dots, \hat{v}_i, \dots, v_n) + \sum_{k=i+1}^n (-1)^{i+k-1} (v_0, \dots, \hat{v}_i, \dots, \hat{v}_k, \dots, v_n) = 0$$

Example (n = 2):
 $\partial_1(\partial_2(v_0, v_1, v_2)) = \partial_1[(v_1, v_2) - (v_0, v_1) + (v_0, v_2)] = v_2 - v_1 - (v_1 - v_0) + v_2 - v_0 = 0$

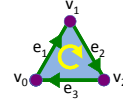
Building blocks for a simplicial complex

0-simplex = vertex = v Grading = dimension

1-simplex = oriented edge = (v₀, v₁)
 Note that the boundary of this edge is v₁ - v₀



2-simplex = oriented face = (v₀, v₁, v₂)
 Note that the boundary of this face is the cycle
 e₁ + e₂ + e₃

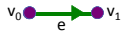


$(v_0, v_1) + (v_1, v_2) - (v_0, v_2)$
 $= (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$

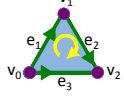
Building blocks for a Δ-complex

0-simplex = vertex = v Grading = dimension

1-simplex = oriented edge = [v₀, v₁]
 Note that the boundary of this edge is v₁ - v₀



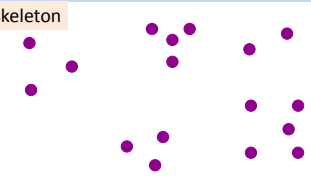
2-simplex = oriented face = [v₀, v₁, v₂]
 Note that the boundary of this face is the cycle
 e₁ + e₂ - e₃



$[v_0, v_1] + [v_1, v_2] - [v_0, v_2]$
 $= [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

Creating a simplicial complex

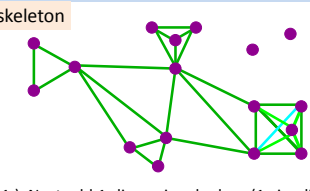
0-skeleton



0.) Start by adding 0-dimensional vertices (0-simplices)

Creating a simplicial complex

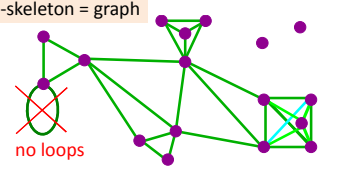
1-skeleton



1.) Next add 1-dimensional edges (1-simplices).
 Note: These edges must connect two vertices.
 I.e., the boundary of an edge is two vertices

Creating a simplicial complex

1-skeleton = graph

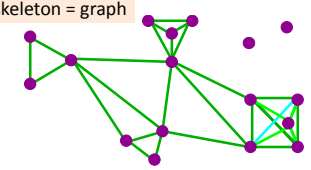


no loops

1.) Next add 1-dimensional edges (1-simplices).
 Note: These edges must connect two vertices.
 I.e., the boundary of an edge is two vertices

Creating a simplicial complex

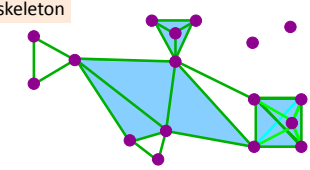
1-skeleton = graph



1.) Next add 1-dimensional edges (1-simplices).
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 I.e., the boundary of an edge is two vertices

Creating a simplicial complex

2-skeleton



2.) Add 2-dimensional triangles (2-simplices).
 Boundary of a triangle = a cycle consisting of 3 edges.

Creating a simplicial complex

3-skeleton

3.) Add 3-dimensional tetrahedrons (3-simplices).
 Boundary of a 3-simplex
 = a cycle consisting of its four 2-dimensional faces.

Creating a simplicial complex

n-skeleton

n.) Add n-dimensional n-simplices, $\{v_1, v_2, \dots, v_{n+1}\}$.
 Boundary of a n-simplex
 = a cycle consisting of (n-1)-simplices.

n-skeleton = $\bigcup_{k \leq n} k$ -simplices

n.) Add n-dimensional n-simplices, $\{v_1, v_2, \dots, v_{n+1}\}$.
 Boundary of a n-simplex
 = a cycle consisting of (n-1)-simplices.

Let $\{v_0, v_1, \dots, v_n\}$ be a simplex.

A subset of $\{v_0, v_1, \dots, v_n\}$ is called a **face** of this simplex.

Ex: The faces of

are $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_1, v_3\}$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$

A simplicial complex K is a set of simplices that satisfies the following conditions:

- Any face of a simplex from K is also in K.
- The intersection of any two simplices in K is either empty or a face of both the simplices.

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simplex = convex hull

valid simplicial complex | invalid simplicial complex

The persistent cosmic web and its filamentary structure I: Theory and implementation - Sousbie, Thierry
<http://inspirehep.net/record/870503/plots>

Triangulations - The Good, The Bad, and The Ugly

Bad: intersection is not a complete side of one of the triangles
Bad: not a triangulation, since the two "triangles" are in fact not triangles at all
Bad: while this does divide the sphere into two triangles, their intersection consists of three sides and vertices, which is not allowed

Good! | Ugly, but a triangulation! | Good!

<http://www.math.cornell.edu/~mec/Winter2009/Victor/part5.htm>

Triangulations - The Good, The Bad, and The Ugly

How many vertices?

Bad: intersection is not a complete side of one of the triangles
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Standard triangulation of the torus:

<https://simomaths.wordpress.com/2013/12/05/from-euler-characteristics-to-cohomology/>

A **simplicial complex** K is a set of simplices that satisfies the following conditions:

- Any face of a simplex from K is also in K .
- The intersection of any two simplices in K is either empty or a face of both the simplices.

Building blocks for an abstract simplicial complex

0-simplex = vertex = $\{v\}$

1-simplex = edge = $\{v_1, v_2\}$

n-simplex = $\{v_0, v_1, \dots, v_n\}$

Let V be a finite set.

A finite **abstract simplicial complex** is a subset A of $P(V)$ such that

- v in V implies $\{v\}$ in A , then
- if X is in A and if $Y \subset X$, then Y is in A

Building blocks for oriented simplicial complex

3-simplex =

$$\begin{aligned} \sigma &= (v_1, v_2, v_3, v_4) = (v_2, v_3, v_1, v_4) = (v_3, v_1, v_2, v_4) \\ &= (v_2, v_1, v_4, v_3) = (v_3, v_2, v_4, v_1) = (v_1, v_3, v_4, v_2) \\ &= (v_4, v_2, v_1, v_3) = (v_4, v_3, v_2, v_1) = (v_4, v_1, v_3, v_2) \\ &= (v_1, v_4, v_2, v_3) = (v_2, v_4, v_3, v_1) = (v_3, v_4, v_1, v_2) \end{aligned}$$

$$\begin{aligned} -\sigma &= (v_2, v_3, v_3, v_4) = (v_3, v_2, v_1, v_4) = (v_1, v_3, v_2, v_4) \\ &= (v_2, v_4, v_1, v_3) = (v_3, v_4, v_2, v_1) = (v_1, v_4, v_3, v_2) \\ &= (v_3, v_2, v_4, v_3) = (v_2, v_3, v_4, v_1) = (v_3, v_1, v_4, v_2) \\ &= (v_4, v_1, v_2, v_3) = (v_4, v_2, v_3, v_1) = (v_4, v_3, v_1, v_2) \end{aligned}$$

Building blocks for a Δ -complex

3-simplex =

$$\sigma = [v_1, v_2, v_3, v_4]$$

Building blocks for a simplicial complex

0-simplex = vertex = v Grading = dimension

1-simplex = oriented edge = (v_0, v_1)

Note that the boundary of this edge is $v_1 - v_0$

2-simplex = oriented face = (v_0, v_1, v_2)

Note that the boundary of this face is the cycle

$$\begin{aligned} e_1 + e_2 + e_3 &= (v_0, v_1) + (v_1, v_2) - (v_0, v_2) \\ &= (v_1, v_2) - (v_0, v_2) + (v_0, v_1) \end{aligned}$$

Building blocks for a Δ -complex

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2-simplex = oriented face = $[v_0, v_1, v_2]$

Note that the boundary of this face is the cycle

$$\begin{aligned} e_1 + e_2 - e_3 &= [v_0, v_1] + [v_1, v_2] - [v_0, v_2] \\ &= [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \end{aligned}$$

$\Delta^n = [v_0, v_1, \dots, v_n]$, $\overset{\circ}{\Delta}^n = \text{interior of } \Delta^n$

A Δ -complex structure on a space X is a collection of maps $\sigma_\alpha: \Delta^n \rightarrow X$, with n depending on the index α , such that:

- The restriction $\sigma_\alpha|_{\overset{\circ}{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\overset{\circ}{\Delta}^n}$.
- Each restriction of σ_α to an $n-1$ face of Δ^n is one of the maps $\sigma_\beta: \Delta^{n-1} \rightarrow X$.

Here we identify the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

- A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α .

Let X be a CW complex.

$X^0 = \text{set of points with discrete topology.}$

Given the $(n-1)$ -skeleton X^{n-1} , form the n -skeleton, X_n , by attaching n -cells via attaching maps $\sigma_\alpha: \partial D_\alpha^n \rightarrow X^{n-1}$, i.e., $X^n = X^{n-1} \cup D_\alpha^n / \sim$ where $x \sim \sigma_\alpha(x)$ for all x in ∂D_α^n

The **characteristic map** $\Phi_\alpha: D_\alpha^n \rightarrow X$ is the map that extends the attaching map $\sigma_\alpha: \partial D_\alpha^n \rightarrow X^{n-1}$ and $\Phi_\alpha|_{D_\alpha^n}$ onto its image is a homeomorphism.



Φ_α is the composition $D_\alpha^n \rightarrow X^{n-1} \cup D_\beta^n \rightarrow X^n \rightarrow X$

Building blocks for a Δ -complex

X^0 = set of points with discrete topology.

Given the $(n-1)$ -skeleton X^{n-1} , form the n -skeleton, X_n , by attaching n -cells via their $(n-1)$ -faces via attaching maps $\sigma_\beta: D^{n-1} \rightarrow X^{n-1}$ such that $\sigma_\beta|_{\partial D^{n-1}}$ is a homeomorphism.



Example: sphere = $\{x \text{ in } \mathbb{R}^3 : ||x|| = 1\}$

Δ -complex  = 

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$$0 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3 \rightarrow 0$$

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Δ -complex  = 

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$$0 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3 \rightarrow 0$$

$H_0 = \mathbb{Z}_0/B_0 = \mathbb{R}^3/\mathbb{R}^2 = \mathbb{R}$

Example: sphere = $\{x \text{ in } \mathbb{R}^3 : ||x|| = 1\}$

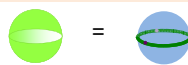

Δ -complex  = 

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$$0 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3 \rightarrow 0$$

$H_1 = \mathbb{Z}_1/B_1 = \mathbb{R}/\mathbb{R} = 0$

Example: sphere = $\{x \text{ in } \mathbb{R}^3 : ||x|| = 1\}$

Δ -complex  = 



$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$$0 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3 \rightarrow 0$$

$H_2 = \mathbb{Z}_2/B_2 = \mathbb{R}/0 = \mathbb{R}$

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Cell complex  = 

Fist image from <http://opencipart.org/detail/1000/a-raised-fist-by-lifarn>


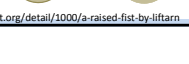
Example: sphere = $\{x \text{ in } \mathbb{R}^3 : ||x|| = 1\}$

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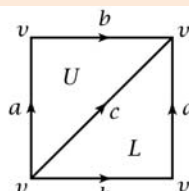
$H_i = \mathbb{Z}_i/B_i = \mathbb{R}/0 = \mathbb{R}$ for $i=0, 2$

$H_i = \mathbb{Z}_i/B_i = 0/0 = \mathbb{R}$ for $i=1, 3, 4, 5, \dots$

Cell complex  = 

Fist image from <http://opencipart.org/detail/1000/a-raised-fist-by-lifarn>

Example: Δ -complex of a Torus



Note the required orientation of edge c for the above complex to be a Δ -complex.
Simplices are oriented via the increasing sequence of their vertices.

Singular homology

A singular n -simplex in a space X is a map $\sigma: \Delta^n \rightarrow X$

These n -simplices form a basis for $C_n(X)$.

$$\partial_n(\sigma) = \sum(-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

Note if X and Y are homeomorphic, then $H_n(X) = H_n(Y)$