Creating a cell complex = CW complex

Building block: n-cells = { x in R^n : || x || ≤ 1 }

Examples: 0-cell = { x in R^0 : ||x|| < 1 } •

1-cell = open interval = { x in R : ||x|| < 1 }

2-cell = open disk = { x in R^2 : ||x|| < 1 }

3-cell = open ball = { x in R^3 : ||x|| < 1 }

Cell complex = CW complex

Building block: n-cells = { x in R^n : || x || < 1 }

Examples: 0-cell = { x in R^0 : ||x|| < 1 } •

1-cell = open interval = { x in R : ||x|| < 1 }

2-cell = open disk = { x in R^2 : ||x|| < 1 }

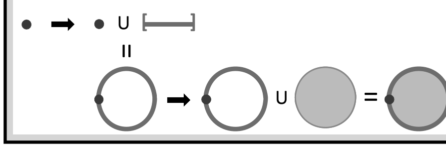
Grading = dimension

∂_n(n-cells) = { x in R^n : || x || = 1 }

Example: disk = { x in R^2 : ||x|| ≤ 1 }

Simplicial complex
3 vertices, 3 edges, 1 triangle

Cell complex
1 vertex, 1 edge,
1 disk.



Cell complex = CW complex

Building block: n-cells = { x in R^n : || x || < 1 }

Examples: 0-cell = { x in R^0 : ||x|| < 1 } •

1-cell = open interval = { x in R : ||x|| < 1 }

2-cell = open disk = { x in R^2 : ||x|| < 1 }

X^0 = set of points with discrete topology.

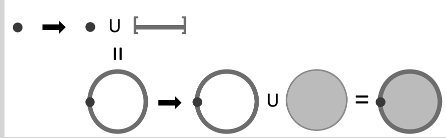
Given the (n-1)-skeleton X^{n-1}, form the n-skeleton, X_n, by attaching n-cells via maps $\sigma_\alpha: \partial D^n \to X^{n-1}$,

I.e., $X^n = X^{n-1} \amalg_\alpha D^n_\alpha / \sim$ where $x \sim \sigma_\alpha(x)$ for all x in ∂D^n_α

Example: disk = { x in R^2 : ||x|| ≤ 1 }

Simplicial complex
3 vertices, 3 edges, 1 triangle

Cell complex
1 vertex, 1 edge,
1 disk.



Example: sphere = { x in R^3 : ||x|| = 1 }

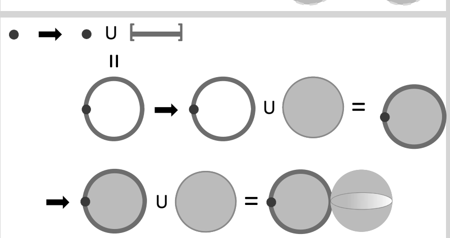
Simplicial complex

Cell complex

Fist image from http://openclipart.org/detail/1000/a-raised-fist-by-liftarn

Example: constant, identity, constant maps

Cell complex
1 vertex, 1 edge,
2 disks.



Let X be a CW complex.

X^0 = set of points with discrete topology.

Given the (n-1)-skeleton X^{n-1}, form the n-skeleton, X_n, by attaching n-cells via attaching maps $\sigma_\alpha: \partial D^n_\alpha \to X^{n-1}$,

I.e., $X^n = X^{n-1} \amalg_\alpha D^n_\alpha / \sim$ where $x \sim \sigma_\alpha(x)$ for all x in ∂D^n_α

The characteristic map $\Phi_\alpha: D^n_\alpha \to X$ is the map that extends the attaching map $\sigma_\alpha: \partial D^n_\alpha \to X^{n-1}$ and $\Phi_\alpha|\mathring{D}^n_\alpha$ onto its image is a homeomorphism.

Φ_α is the composition $D^n_\alpha \to X^{n-1} \amalg_\beta D^n_\beta \to X^n \to X$

Your name homology

3 ingredients:

1.) Objects

2.) Grading

3.) Boundary map

Grading

Grading: Each object is assigned a unique grade.

Let $X_n = \{x_1, \ldots, x_k\}$ = generators of grade n.

Extend grading on the set of generators to the set of n-chains: C_n = set of n-chains = $R[X_n]$

Normally n-chains in C_n are assigned to the grade n.

$\partial_n : C_n \rightarrow C_{n-1}$ such that $\partial^2 = 0$

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \ldots \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$H_n = Z_n/B_n$ = (kernel of ∂_n)/ (image of ∂_{n+1})

$$= \frac{\text{null space of } M_n}{\text{column space of } M_{n+1}}$$

Rank H_n = Rank Z_n – Rank B_n

Čech homology

Given $\bigcup_{\alpha \text{ in } A} V_\alpha$ where V_α open for all α in A.

Objects = finite intersections = $\{ \bigcap_{i=1}^{n} V_{\alpha_i} : \alpha_i \text{ in } A \}$

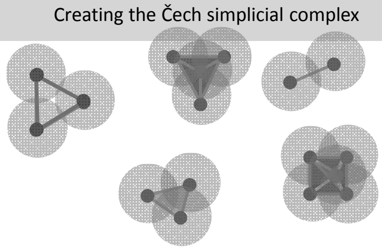
Grading = n = depth of intersection.

$$\partial_{n+1} \left(\bigcap_{i=1}^{n} V_{\alpha_i} \right) = \sum_{j=1}^{n} \left(\bigcap_{\substack{i=1 \\ i \neq j}}^{n} V_{i\alpha} \right)$$

Ex: $\partial_0(V_\alpha) = 0$, $\partial_1(V_\alpha \cap V_\beta) = V_\alpha + V_\beta$

$\partial_2(V_\alpha \cap V_\beta \cap V_\gamma) = (V_\alpha \cap V_\beta) + (V_\alpha \cap V_\gamma) + (V_\beta \cap V_\gamma)$

Creating the Čech simplicial complex



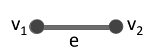
1.) $B_1 \cap \ldots \cap B_{k+1} \neq \emptyset$, create k-simplex $\{v_1, \ldots, v_{k+1}\}$.

Unoriented simplicial complex using $\mathbf{Z_2}$ coefficients

Grading = dimension

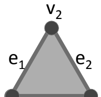
0-simplex = vertex = v ●

1-simplex = edge = $\{v_1, v_2\}$



Note that the boundary of this edge is $v_2 + v_1$

2-simplex = face = $\{v_1, v_2, v_3\}$

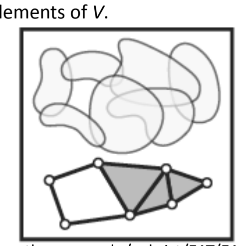


Note that the boundary of this face is the cycle

$e_1 + e_2 + e_3$

$= \{v_1, v_2\} + \{v_2, v_3\} + \{v_1, v_3\}$

Nerve Lemma: If *V* is a finite collection of subsets of X with all non-empty intersections of subcollections of *V* contractible, then N(V) is homotopic to the union of elements of *V*.



http://www.math.upenn.edu/~ghrist/EAT/EATchapter2.pdf

Theorem: The choice of triangulation does not affect the homology.

$\partial_n : C_n \rightarrow C_{n-1}$ such that $\partial^2 = 0$

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \ldots \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$H_n = Z_n/B_n$ = (kernel of ∂_n)/ (image of ∂_{n+1})

$$= \frac{\text{null space of } M_n}{\text{column space of } M_{n+1}}$$

Rank H_n = Rank Z_n – Rank B_n

Building blocks for oriented simplicial complex

3-simplex =

$\sigma = (v_1, v_2, v_3, v_4) = (v_2, v_3, v_1, v_4) = (v_3, v_1, v_2, v_4)$
$= (v_2, v_1, v_4, v_3) = (v_3, v_2, v_4, v_1) = (v_1, v_3, v_4, v_2)$
$= (v_4, v_2, v_1, v_3) = (v_4, v_3, v_2, v_1) = (v_4, v_1, v_3, v_2)$
$= (v_1, v_4, v_2, v_3) = (v_2, v_4, v_3, v_1) = (v_3, v_4, v_1, v_2)$

$-\sigma = (v_2, v_1, v_3, v_4) = (v_3, v_2, v_1, v_4) = (v_1, v_3, v_2, v_4)$
$= (v_2, v_4, v_1, v_3) = (v_3, v_4, v_2, v_1) = (v_1, v_4, v_3, v_2)$
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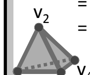


Čech homology

Given $\bigcup_{\alpha \text{ in } A} V_\alpha$ where V_α open for all α in A.

Objects = finite intersections = $\{ \bigcap_{i=1}^{n} V_{\alpha_i} : \alpha_i \text{ in } A \}$

Grading = n = depth of intersection.

$$\partial_{n+1} \left(\bigcap_{i=1}^{n} V_{\alpha_i} \right) = \sum_{j=1}^{n} \left(\bigcap_{\substack{i=1 \\ i \neq j}}^{n} V_{i\alpha} \right)$$

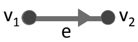
Ex: $\partial_0(V_\alpha) = 0$, $\partial_1(V_\alpha \cap V_\beta) = V_\alpha + V_\beta$

$\partial_2(V_\alpha \cap V_\beta \cap V_\gamma) = (V_\alpha \cap V_\beta) + (V_\alpha \cap V_\gamma) + (V_\beta \cap V_\gamma)$

Oriented simplicial complex

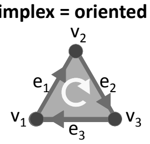
0-simplex = vertex = v ● Grading = dimension

1-simplex = oriented edge = (v_1, v_2)



Note that the boundary of this edge is $v_2 - v_1$

2-simplex = oriented face = (v_1, v_2, v_3)



Note that the boundary of this face is the cycle

$e_1 + e_2 + e_3$

$= (v_1, v_2) + (v_2, v_3) - (v_1, v_3)$

$= (v_1, v_2) - (v_1, v_3) + (v_2, v_3)$

$$\partial_n(v_0, \ldots, v_n) = \sum_{i=0}^{n}(-1)^i(v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n) \quad \text{where } \widehat{v_i} \text{ means omit } v_i$$

$$\partial_{n-1}(\partial_n(v_0, \ldots, v_n)) = \partial_{n-1}\Big(\sum_{i=0}^{n}(-1)^i(v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n)\Big)$$

$$= \sum_{i=0}^{n}(-1)^i\partial_{n-1}(v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n)$$

$$= \sum_{i=0}^{n}(-1)^i\left[\sum_{k=0}^{i-1}(-1)^k(v_0, \ldots, \widehat{v_k}, \ldots, \widehat{v_i}, \ldots, v_n) + \sum_{k=i+1}^{n}(-1)^{k-1}(v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_k}, \ldots, v_n)\right]$$

$$= \sum_{k<i}(-1)^{i+k}(v_0, \ldots, \widehat{v_k}, \ldots, \widehat{v_i}, \ldots, v_n) + \sum_{k>i}(-1)^{i+k-1}(v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_k}, \ldots, v_n) = 0$$

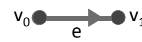
Example ($n = 2$):

$$\partial_1(\partial_2(v_0, v_1, v_2)) = \partial_1[(v_1, v_2) - (v_0, v_1) + (v_0, v_2)] = v_2 - v_1 - (v_1 - v_0) + v_2 - v_0 = 0$$

Building blocks for a simplicial complex

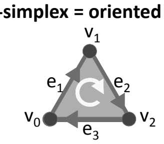
0-simplex = vertex = v ● Grading = dimension

1-simplex = oriented edge = (v_0, v_1)



Note that the boundary of this edge is $v_1 - v_0$

2-simplex = oriented face = (v_0, v_1, v_2)



Note that the boundary of this face is the cycle

$e_1 + e_2 + e_3$

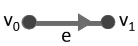
$= (v_0, v_1) + (v_1, v_2) - (v_0, v_2)$

$= (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$

Building blocks for a Δ-complex

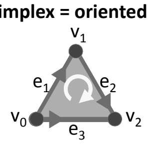
0-simplex = vertex = v ● Grading = dimension

1-simplex = oriented edge = $[v_0, v_1]$



Note that the boundary of this edge is $v_1 - v_0$

2-simplex = oriented face = $[v_0, v_1, v_2]$



Note that the boundary of this face is the cycle

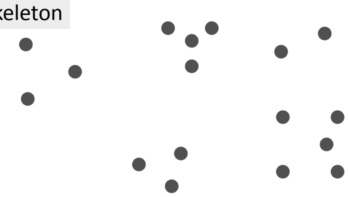
$e_1 + e_2 - e_3$

$= [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$

$= [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

Creating a simplicial complex

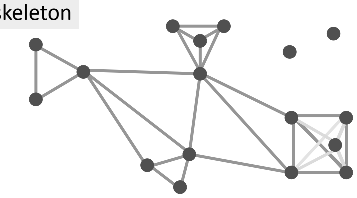
0-skeleton



0.) Start by adding 0-dimensional vertices (0-simplices)

Creating a simplicial complex

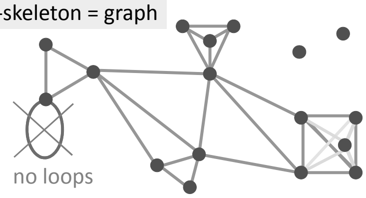
1-skeleton



1.) Next add 1-dimensional edges (1-simplices).
Note: These edges must connect two vertices.
I.e., the boundary of an edge is two vertices

Creating a simplicial complex

1-skeleton = graph



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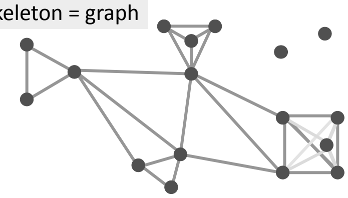
Creating a simplicial complex

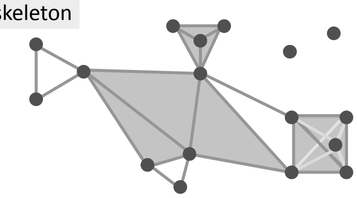
1-skeleton = graph



1.) Next add 1-dimensional edges (1-simplices).
Note: These edges must connect two vertices.
I.e., the boundary of an edge is two vertices

Creating a simplicial complex

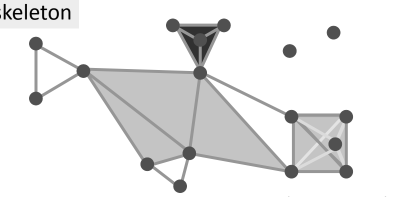
2-skeleton



2.) Add 2-dimensional triangles (2-simplices).
Boundary of a triangle = a cycle consisting of 3 edges.

Creating a simplicial complex

3-skeleton



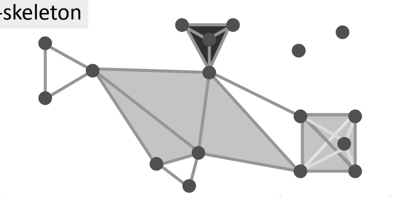
3.) Add 3-dimensional tetrahedrons (3-simplices).

Boundary of a 3-simplex

= a cycle consisting of its four 2-dimensional faces.

Creating a simplicial complex

n-skeleton

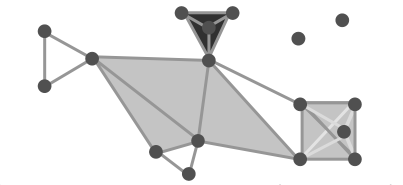


n.) Add n-dimensional n-simplices, $\{v_1, v_2, \ldots, v_{n+1}\}$.

Boundary of a n-simplex

= a cycle consisting of (n-1)-simplices.

n-skeleton $= \bigcup_{k \leq n}$ k-simplices



n.) Add n-dimensional n-simplices, $\{v_1, v_2, \ldots, v_{n+1}\}$.

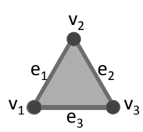
Boundary of a n-simplex

= a cycle consisting of (n-1)-simplices.

Let $\{v_0, v_1, \ldots, v_n\}$ be a simplex.

A subset of $\{v_0, v_1, \ldots, v_n\}$ is called a face of this simplex.

Ex: The faces of

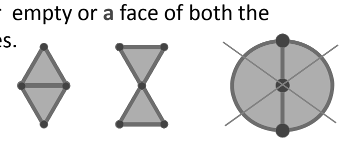


are $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_1, v_3\}$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$

A simplicial complex K is a set of simplices that satisfies the following conditions:

1. Any face of a simplex from K is also in K.

2. The intersection of any two simplices in K is either empty or a face of both the simplices.



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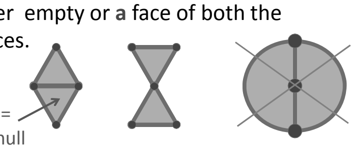
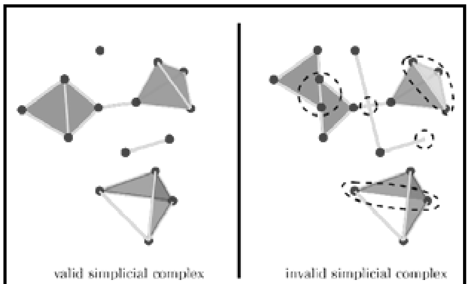
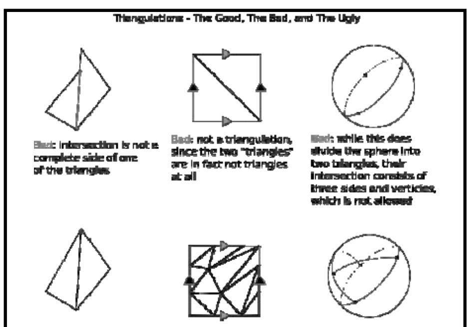
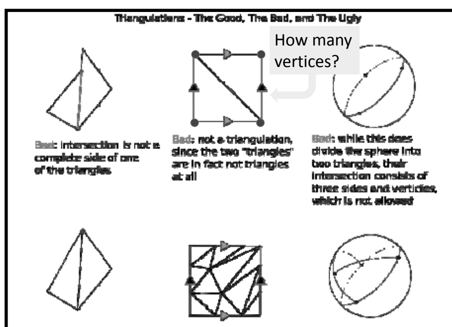




The persistent cosmic web and its filamentary structure I: Theory and implementation - Sousbie, Thierry http://inspirehep.net/record/870503/plots

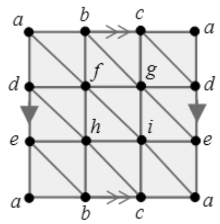


http://www.math.cornell.edu/~mec/Winter2009/Victor/part5.htm



http://www.math.cornell.edu/~mec/Winter2009/Victor/part5.htm

Standard triangulation of the torus:



https://simomaths.wordpress.com/2013/12/05/from-euler-characteristics-to-cohomology-ii/

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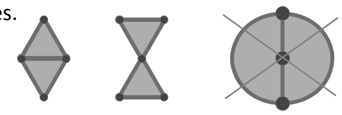

Building blocks for an abstract simplicial complex

0-simplex = vertex = {v}

1-simplex = edge = $\{v_1, v_2\}$

n-simplex = $\{v_0, v_1, \ldots, v_n\}$

Let V be a finite set.

A finite abstract simplicial complex is a subset A of P(V) such that
1.) v in V implies {v} in A, then
2.) if X is in A and if $Y \subset X$, then Y is in A

Building blocks for oriented simplicial complex

3-simplex =

$\sigma = (v_1, v_2, v_3, v_4) = (v_2, v_3, v_1, v_4) = (v_3, v_1, v_2, v_4)$
$= (v_2, v_1, v_4, v_3) = (v_3, v_2, v_4, v_1) = (v_1, v_3, v_4, v_2)$
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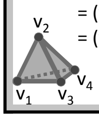
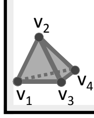
$-\sigma = (v_2, v_1, v_3, v_4) = (v_3, v_2, v_1, v_4) = (v_1, v_3, v_2, v_4)$
$= (v_2, v_4, v_1, v_3) = (v_3, v_4, v_2, v_1) = (v_1, v_4, v_3, v_2)$
$= (v_1, v_2, v_4, v_3) = (v_2, v_3, v_4, v_1) = (v_3, v_1, v_4, v_2)$
$= (v_4, v_1, v_2, v_3) = (v_4, v_2, v_3, v_1) = (v_4, v_3, v_1, v_2)$



Building blocks for a Δ-complex

3-simplex =

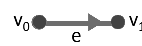
$\sigma = [v_1, v_2, v_3, v_4]$



Building blocks for a simplicial complex

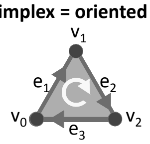
0-simplex = vertex = v ● Grading = dimension

1-simplex = oriented edge = (v_0, v_1)



Note that the boundary of this edge is $v_1 - v_0$

2-simplex = oriented face = (v_0, v_1, v_2)



Note that the boundary of this face is the cycle

$e_1 + e_2 + e_3$
$= (v_0, v_1) + (v_1, v_2) - (v_0, v_2)$
$= (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$

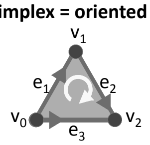
Building blocks for a Δ-complex

0-simplex = vertex = v ● Grading = dimension

1-simplex = oriented edge = $[v_0, v_1]$

Note that the boundary of this edge is $v_1 - v_0$

2-simplex = oriented face = $[v_0, v_1, v_2]$



Note that the boundary of this face is the cycle

$e_1 + e_2 - e_3$
$= [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$
$= [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

$\Delta^n = [v_0, v_1, \ldots, v_n]$, $\mathring{\Delta}^n$ = interior of Δ^n

A Δ-complex structure on a space X is a collection of maps $\sigma_\alpha: \Delta^n \to X$, with n depending on the index α, such that:

(i) The restriction $\sigma_\alpha|\mathring{\Delta}^n$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|\mathring{\Delta}^n$.

(ii) Each restriction of σ_α to an n-1 face of Δ^n is one of the maps $\sigma_\beta: \Delta^{n-1} \to X$.

Here we identify the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

(iii) A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

Let X be a CW complex.

X^0 = set of points with discrete topology.

Given the (n-1)-skeleton X^{n-1}, form the n-skeleton, X_n, by attaching n-cells via attaching maps $\sigma_\alpha: \partial D^n_\alpha \to X^{n-1}$,

I.e., $X^n = X^{n-1} \sqcup_\alpha D^n_\alpha / \sim$ where $x \sim \sigma_\alpha(x)$ for all x in ∂D^n_α

The characteristic map Φ_α: $D^n_\alpha \to X$ is the map that extends the attaching map σ_α: $\partial D^n_\alpha \to X^{n-1}$ and $\Phi_\alpha|\mathring{D}^n_\alpha$ onto its image is a homeomorphism.

Φ_α is the composition $D^n_\alpha \to X^{n-1} \sqcup_\beta D^n_\beta \to X^n \to X$

Building blocks for a Δ-complex

X^0 = set of points with discrete topology.

Given the (n-1)-skeleton X^{n-1}, form the n-skeleton, X_n, by attaching n-cells via their (n-1)-faces via

attaching maps $\sigma_\beta: D^{n-1} \to X^{n-1}$ such that

$\sigma_\beta|\mathring{D}^{n-1}$ is a homeomorphism.

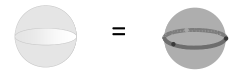
Example: sphere = { x in R^3 : ||x|| = 1 }

Δ-complex 

$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

$0 \to R^2 \to R^3 \to R^3 \to 0$

Example: sphere = { x in R^3 : ||x|| = 1 }

Δ-complex 

$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

$0 \to R^2 \to R^3 \to R^3 \to 0$

$H_0 = Z_0/B_0 = R^3/R^2 = R$

Example: sphere = { x in R^3 : ||x|| = 1 }

Δ-complex 

$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

$0 \to R^2 \to R^3 \to R^3 \to 0$

$H_1 = Z_1/B_1 = R/R = 0$

Example: sphere = { x in R^3 : ||x|| = 1 }

Δ-complex 

$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

$0 \to R^2 \to R^3 \to R^3 \to 0$

$H_2 = Z_2/B_2 = R/0 = R$

Example: sphere = { x in R^3 : ||x|| = 1 }

$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

Cell complex 

Fist image from http://openclipart.org/detail/1000/a-raised-fist-by-liftarn

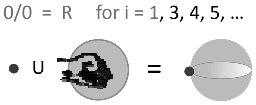
Example: sphere = { x in R^3 : ||x|| = 1 }

$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

$0 \to R \to 0 \to R \to 0$

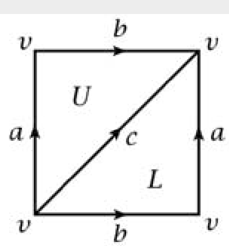
$H_i = Z_i/B_i = R/0 = R$ for i = 0, 2

$H_i = Z_i/B_i = 0/0 = R$ for i = 1, 3, 4, 5, …

Cell complex 

Fist image from http://openclipart.org/detail/1000/a-raised-fist-by-liftarn

Example: Δ-complex of a Torus



Note the **required** orientation of edge c for the above complex to be a Δ-complex.
Simplices are oriented via the increasing sequence of their vertices.

Singular homology

A singular n-simplex in a space X is a map

$\sigma: \Delta^n \to X$

These n-simplices form a basis for $C_n(X)$.

$\partial_n(\sigma) = \Sigma(-1)^i \sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n]$

Note if X and Y are homeomorphic, then

$H_n(X) = H_n(Y)$