Theorem 2.2 (Havel-Hakimi): Consider a list $s = [d_1, d_2, ..., d_n]$ of n numbers in descending (non-increasing) order. This list is graphic if and only if $s^* = [d_1^*, d_2^*, ..., d_{n-1}^*]$ of $n-1$ numbers is graphic as well, where

$$d_i^* = \begin{cases} d_{i+1} - 1 & \text{for i = 1, 2, . . . , } d_1 \\ d_{i+1} & \text{otherwise} \end{cases}$$

I.e., s is graphic iff s^* is graphic.

Proof: Let $s = [d_1, d_2, ..., d_n]$ where $d_i \geq d_{i+1}$, $d_i \in \{0, 1, 2, ...\}$

Let $s^* = [d_1^*, d_2^*, ..., d_{n-1}^*]$ where

$$d_i^* = \begin{cases} d_{i+1} - 1 & \text{for i = 1, 2, . . . , } d_1 \\ d_{i+1} & \text{otherwise} \end{cases}$$

$$\text{Thus } d_i = \begin{cases} d_1 & \text{i = 1} \\ d_{i-1}^* + 1 & \text{for i = 2, 3, . . . , } d_1 + 1 \\ d_{i-1}^* & \text{otherwise} \end{cases}$$

($\Leftarrow$) Suppose s^* is graphic.

Claim s is graphic.

s^* graphic implies $\exists$ simple graph G^* with degree sequence s^*.

Let $G^* = (V^*, E^*)$ where $V^* = \{v_1, ..., v_{n-1}\}$ and $\delta(v_i) = d_i^*$.

Let $V = \{u, v_1, ..., v_{n-1}\}$

Let $E = E^* \cup \{< u, v_i > \ | \ i = 1, ..., d_1\}$

$$\text{Then } \delta(w) = \begin{cases} d_1 & w = u \\ d_i^* + 1 & w = v_i, \ i = 1, ..., d_1 \\ d_i^* & \text{else} \end{cases} .$$

Thus $G = (V, E)$ has degree sequence s. Note G is a simple graph. Thus s is graphic.

($\Rightarrow$) Suppose s is graphic.

Claim s^* is graphic.

s graphic implies $\exists$ simple graph $G = (V, E)$ with degree sequence s, where $V = \{v_1, ..., v_n\}$ where $\delta(v_i) = d_i$.

Let $N(v_1) = \{w \in V \mid \ < v_1, w > \in E\}$

Let $A_G = N(v_1) \cap \{v_2, ..., v_{d_1+1}\}$

Note $0 \leq |A_G| \leq d_1$.

Among all simple graphs G with degree sequence s, choose one such that $|A_G|$ is as large as possible.

Let $G^* = G - v_1 = (V^*, E^*)$ where $V^* = \{v_2, ..., v_n\}$ $\blacksquare$ and $E^* = \{e \in E \mid e$ is not adjacent to $v_i\}$

Note G^* is a simple graph.

Case 1: Suppose $|A_G| = d_1$. Then v_1 is adjacent to $v_2, ..., v_{d_1+1}$

Then G^* has degree sequence s^*. Thus s^* is graphic. $\blacksquare$

Case 2: Suppose $|A_G| < d_1$.

Then $\exists v_j,\ j \in \{2, ..., d_1 + 1\}$ such that v_1 is not adjacent to v_j in G and $\exists v_\ell,\ \ell > d_1 + 1$ such that v_1 is adjacent to v_ℓ in G

Note $j \leq d_1 + 1 < \ell$. Thus $\delta(v_j) \geq \delta(v_\ell)$.

Case 2a: Suppose $\delta(v_j) = \delta(v_\ell)$.

Then relabel the vertices v_j and v_ℓ. But by relabeling, we increase $|A_G|$ which contradicts maximality, so case 2a cannot occur.

Case 2b: Suppose $\delta(v_j) > \delta(v_\ell)$.

Then $\exists x \in V$ such that
$< v_j, x > \in E(G)$, but $< v_\ell, x > \notin E(G^*)$

Let $G' = (V, E')$ where $V(G') = V(G)$ and

$$E' = [E(G) - \{< v_j, x >, < v_\ell, v_1 >\}] \cup \{< v_j, v_1 >, < v_\ell, x >\}$$

Note G' is a simple graph with degree sequence s. But $|A'_G| = |A_G| + 1$, contradicting maximality, so case 2b cannot occur.

Thus case 1 holds and s^* is graphic.