Thm 4.6: Let G be a graph with n vertices. Suppose also that $\exists$ non-adjacent vertices u and w such that $\delta(u) + \delta(w) \geq n$.

Then G is Hamiltonian $\iff G+ < u, w >$ is Hamiltonian.

Proof ($\Rightarrow$) A Hamiltonian cycle in G is also a Hamiltonian cycle in $G+ < u, w >$.

($\Leftarrow$) Suppose $G+ < u, w >$ is Hamiltonian.

Proof by contradiction: Suppose G is not Hamiltonian.

Thus the Hamiltonian cycle in $G+ < u, w >$ must contain the edge $< u, w >$ (else this cycle would be a Hamiltonian cycle in G).

Thus we can write the Hamiltonian cycle as

$$u, w, v_1, v_2, \ldots, v_{n-2}, u$$

Suppose $< u, v_{i-1} >$ and $< w, v_i >$ are both edges in G. Then

$$w, v_i, \ldots, v_{n-2}, u, v_{i-1}, v_{i-2}, \ldots, v_1, w$$

is a Hamiltonian cycle in G, a contradiction.

Thus $< u, v_{i-1} > \in E(G)$ implies $< w, v_i > \notin E(G)$.

Suppose in G, $\delta(u) = k$, then $\delta(w) \leq$

Thus in G, $\delta(u) + \delta(w) \leq$

Defn 4.3: If $V(G) = n$, the **closure** of G is the graph obtained from G by iteratively adding edges to G joining non-adjacent vertices u and w where $\delta(u) + \delta(w) \geq n$.

To create the closure of G, create the sequence

$$G_0 = G,\ G_1 = G_0 + < u_0, w_0 >, \ldots, \\ G_k = G_{k-1} + < u_{k-1}, w_{k-1} >$$

where $< u_i, w_i > \notin G_i$ and $\delta(u_i) + \delta(w_i) \geq n$ in G_i.

Moreover if $\delta(u) + \delta(w) \geq n$ in G_k, then $< u, w > \in G_k$.

Thm 4.7: A simple graph is Hamiltonian if and only if its closure is Hamiltonian.

Thm (Ore 1960).
If G is a simple graph with $|V(G)| = n \geq 3$ and if $\forall$ non-adjacent vertices u and w, $\delta(u) + \delta(w) \geq n$.

Then G is Hamiltonian.

Proof: The closure of G is and is Hamiltonian.

Corollary 4.5: If G is a simple graph with $|V(G)| = n \geq 3$ and each vertex v has degree $\delta(v) \geq \frac{n}{2}$, then G is Hamiltonian.