The median for this midterm was 72.25. This isn't bad considering how many of you did not know definitions. I expect grades to improve on the final exam if you learn definitions (make flashcards) and follow (proof) writing advice. Note I do take improvement into consideration as discussed in class and on ICON.

It is easy to make (false) assumptions. What you learn in this and other math classes will hopefully help you make fewer assumptions and to be more precise. The final exam will have a similar format to this midterm, so that you can do extra problems in case you make an incorrect assumption in other problem(s).

Description of grading scheme and advice for final exam (and HW, quizzes, etc).

For problem #2, I did allow many/most of you to get away with very poor writing. You should keep in mind that other professors as well as the real world may grade you more harshly. Poor writing can result in costly misunderstandings, so working on improving your writing skills (both English and Math) is highly recommended. But for this class, we will try to grade your English gently (but if we can't infer what you mean, we can't give you points).

For definitions, you must be complete and precise. You can use English and/or math notation, but your definition must be accurate. Thus I recommend flash cards for definitions. As you go through the flash cards, think about examples as well as various parts of the definition and why the parts are included in the definition.

If you don't know definitions, you will likely make mistakes in proofs and in modeling. Knowing definitions gives you more tools to model real life problems.

Note for problems 3 and 4, knowing definitions and theorems can be very helpful when creating examples. For problem 3, some of you gave an example of two non-isomorphic graphs that satisfied your incorrect definition of isomorphic (if two graphs have the same degree sequence, then there exists a bijection between the vertices and between the edges of the two graphs).

For problem 5, note I gave 2 proofs. The second proof uses more notations for describing a walk and thus might be easier to write/read.

I also gave 2 proofs for problem 6, the induction proof. Note you could earn more than half the points, by proving a base case, stating the induction hypothesis, and starting the proof that S(m) implies S(m+1) for proof 1 or S($\leq$m) implies S(m+1) for proof 2. Note that a graph needs to be connected in order to apply the induction hypothesis. For more on induction proofs, please see

http://homepage.divms.uiowa.edu/ idarcy/COURSES/4060/induction_cycle.pdf

Note the last problem, #7, was the "easiest" as it followed from definitions (and if you forgot the precise definition for k-vertex colorable, you could modify the definition of k-edge colorable).

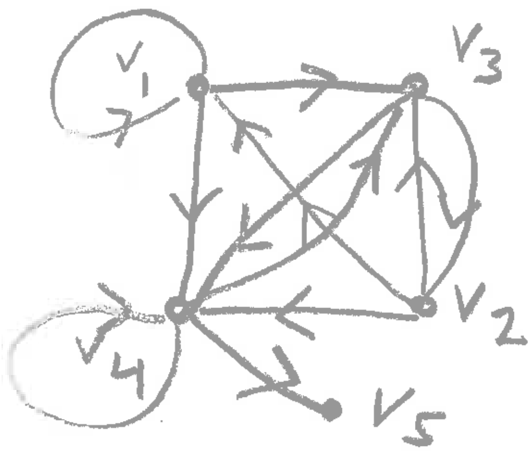
Math 4060 Midterm March 8, 2019 SHOW ALL WORK

Choose **5** from the following 7 problems. **Clearly indicate your choices.** You may attempt all problems for additional partial credit as discussed in class. Note each problem is worth 20 points.

Circle the numbers of the 5 problems you would like graded.

1 2 3 4 5 6 7

1a.) Draw the directed graph with adjaceny matrix: $\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$



1b.) If v_i is the vertex corresponding to the ith column, then

$\delta_{in}(v_1) =$ 2

$N_{in}(v_1) =$ $\{v_2\}$

$\delta_{out}(v_1) =$ 3

$N_{out}(v_1) =$ $\{v_3, v_4\}$

A directed walk between v_1 and v_2 is

$$v_1, v_3, v_2$$

$$\text{or} \quad v_1, \overrightarrow{v_1, v_3}, v_3, \overrightarrow{v_3, v_2}, v_2$$

Is G weakly connected? *yes*

Is G strongly connected? *no*

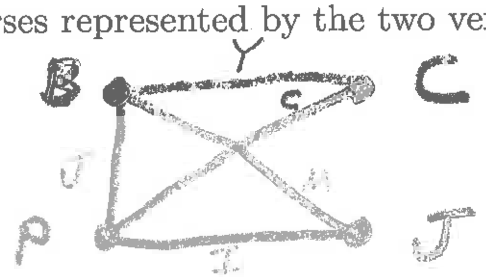
2.) Describe how to mathematically model the following problem.

The following table shows classes taken by 5 students where an x in row i, column j means student i is taking class j. What is the minimum number of time slots for final exams (i.e., final exam times), so that no student has 2 final exams scheduled for the same time.

	Bio	Calc	Java	Physics
Claus		x		x
Irma			x	x
Javier	x			x
Mariel	x		x	
Yuanan	x	x		

(a) Draw the graph that can be used to solve this problem. What do your vertices represent? What do your edges represent?

Vertices represent courses. Edges represent a conflict for a final exam time or equivalently an edge is drawn between two vertices if their is at least one student taking the two courses represented by the two vertices.



(b) State and solve the relevant graph theory problem.

Find $\chi(G)$, the chromatic number of G.

Since G contains an odd cycle, $\chi(G) > 2$. By the coloring below, $\chi(G) \leq 3$. Thus $\chi(G) = 3$

(c) Explain how the solution found in part (b) can be applied to the problem you are modeling.

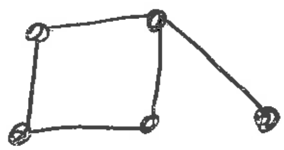
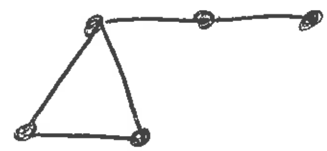
Each color corresponds to a time slot. Since $\chi(G) = 3$, 3 time slots are needed for the final exams so that no student has 2 final exams scheduled for the same time.

3.) Define: Two graphs G and H are isomorphic if

$\exists$ bijection $f : V(G) \to V(H)$ which induces a bijection $f' : E(G) \to E(H)$ such that $f'(< u, v >) =< f(u), f(v) >$

Give an example of two simple connected non-isomorphic graphs with the same degree sequence. Justify your answer. Note if your example does not satisfy all the requested conditions, please state which conditions are missing for partial credit.

Example 1: Degree sequence $[3, 2, 2, 2, 1]$

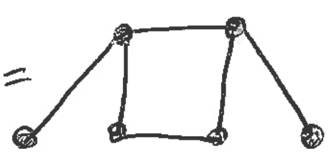
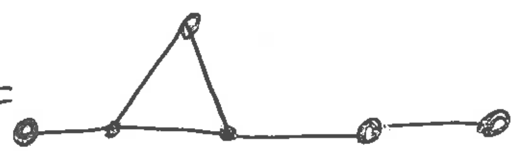
$G =$  $H =$ 

Note that G has a cycle of length 4, but H does not have a cycle of length 4.

Alternatively, note the degree 1 vertex of H has a neighbor that has degree 2, while the degree 1 vertex of G has a neighbor with degree 3.

Alternatively, G is bipartite, while H is not bipartite.

Example 2: Degree sequence $[3, 3, 2, 2, 1, 1]$

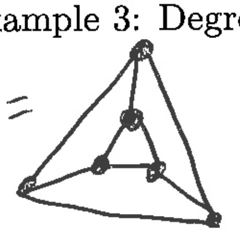
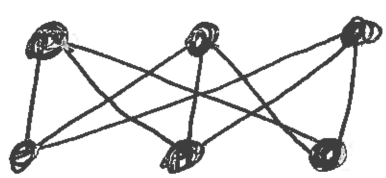
$G =$  $H =$ 

Note that G has a cycle of length 4, but H does not have a cycle of length 4.

Alternatively, H has a degree 1 vertex with neighbor that has degree 2, while the degree 1 vertices of G both have a neighbor with degree 3.

Alternatively, G is bipartite, while H is not bipartite.

Example 3: Degree sequence $[3, 3, 3, 3, 3, 3]$

$G =$  $H =$ 

Note that G has 2 cycles of length 3, but H does not have a cycle of length 3.

Alternatively, G is planar, while H is not planar.

Alternatively, G is not bipartite, while H is bipartite.

4.) Define: The chromatic number, $\chi(G) = \min\{k \mid G \text{ is } k\text{-vertex colorable }\}$.

Give an example of a non-planar graph, G, with $\kappa(G) = 1$, $\lambda(G) = 2$, $\Delta(G) = 5$, and $\chi(G) = 2$. Justify your answer. Note if your example does not satisfy all the requested conditions, please state which conditions are missing for partial credit.

$\chi(G) = 2$ implies G is bipartite. We want a bipartite non-planar graph. Note $K_{3,3}$ is a non-planar and bipartite (alternatively, could try a subdivision of K_5 or $K_{3,3}$).

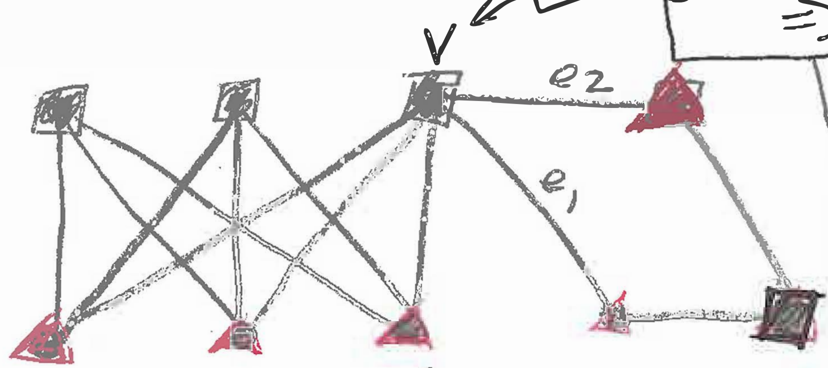
If we start with $K_{3,3}$ we can add two new neighbors to one of the vertices, v to create a graph with $\Delta(G) = 5$.

$\kappa(K_{3,3}) = 3$ and $\lambda(K_{3,3}) = 3$, so will try to make v a cut vertex of G.

Since $\lambda(G) = 2$, every pair of vertices should lie on a cycle. Since G is bipartite, all cycles have even length. Thus we can put the new neighbors of w on a cycle of length 4 (alternatively, one can let G be a multi-graph).

$\delta(v) = 5$, largest degree
$\Rightarrow \Delta(G) = 5$

v is a cut vertex
$\Rightarrow \kappa(G) = 1$



contains
$K_{3,3}$
Thus non planar

$\{e_1, e_2\}$ is an edge cut
$\Rightarrow \lambda(G) \leq 2$
~~All cycles even length~~
Every edge in cycle
$\Rightarrow \lambda(G) \geq 2$
Thus $\lambda(G) = 2$

Coloring of vertices
$\boxed{}$ or $\triangle$
$\Rightarrow \chi(G) = 2$

5.) Define: A directed path between vertices u and v is a directed walk between vertices u and v where no vertex is repeated.

I.e, $\exists$ a sequence of vertices $u = u_1, u_2, ..., u_{n-1}, u_n = v$ such that $< \overrightarrow{u_i, u_{i+1}} >$ is an edge in the digraph $\forall i = 1, ..., n-1$. and $u_i \neq u_j \forall i \neq j$.

I.e, $\exists$ a sequence alternating between vertices and edges in the digraph such that $u = u_1, < \overrightarrow{u_1, u_2} >, u_2, < \overrightarrow{u_3, u_4} >, ..., u_{n-1}, < \overrightarrow{u_{n-1}, u_n} >, u_n = v$ such that $u_i \neq u_j \forall i \neq j$.

Prove that if there is a directed walk between vertices u and v, then there is a directed path between vertices u and v.

Note: This is similar but not the same as HW problem where you had to find a path which is a portion of the walk. You can modify the proof below to include this restriction by applying it to the subgraph which consists of only the vertices and arcs in the directed walk between vertices u and v.

Proof: Let $w_0, w_1, ..., w_{k-1}, w_k$ be a shortest directed walk between u and v where $w_0 = u$ and $w_k = v$. Note a shortest directed walk exists between u and v since we know that there is a directed walk between u and v.

Suppose $w_0, w_1, ..., w_{k-1}, w_k$ is not a path. Then $\exists i, j$ such that $i < j$ and $w_i = w_j$ (i.e, the walk encounters the same vertex twice).
Since $w_i = w_j$, $< \overrightarrow{w_i, w_{j+1}} > = < \overrightarrow{w_j, w_{j+1}} >$ is an edge in the digraph.

Then $w_0, w_1, ..., w_i, w_{j+1}, w_{k-1}, w_k$ is a shorter directed walk between u and v, contradicting that $w_0, w_1, ..., w_{k-1}, w_k$ is a shortest directed walk between u and v

Note: there can be more than 1 shortest path between vertices, so I used "a" shortest instead of "the" shortest.

Alternate proof Note sometimes using more notation can be helpful. In the proof below, it is more obvious that we didn't accidently include an "arc" that doesn't actually exist in our shortened walk.

Let $w_0, < \overrightarrow{w_0, w_1} >, w_1, ..., w_{k-1}, < \overrightarrow{w_{k-1}, w_k} >, w_k$ be a shortest directed walk between u and v where $w_0 = u$ and $w_k = v$. Note a shortest directed walk exists between u and v since we know that there is a directed walk between u and v.

Suppose $w_0, < \overrightarrow{w_0, w_1} >, w_1, ..., w_{k-1}, < \overrightarrow{w_{k-1}, w_k} >, w_k$ is not a path. Then $\exists i, j$ such that $i < j$ and $w_i = w_j$ (i.e, the walk encounters the same vertex twice).

Since $w_i = w_j$, $< \overrightarrow{w_i, w_{j+1}} > = < \overrightarrow{w_j, w_{j+1}} >$ is an edge in the digraph.

Thus $w_0, < \overrightarrow{w_0, w_1} >, w_1, ..., w_i, < \overrightarrow{w_j, w_{j+1}} >, w_{j+1}, ..., w_{k-1}, < \overrightarrow{w_{k-1}, w_k} >, w_k$ is a shorter directed walk between u and v, contradicting that $w_0, w_1, ..., w_{k-1}, w_k$ is a shortest directed walk between u and v

6.) Define: T is a subgraph of G if

$V(T) \subset V(G)$, $E(T) \subset E(G)$, and $\{v \mid \exists < u, v > \in E(T)\} \subset V(T)$ (i.e., the endpoints of edges in T are vertices in T).

Let G be a simple connected graph. Use induction on $|E(G)|$ to prove that G contains a subgraph T where T is a tree and $V(T) = V(G)$.

Proof by induction on $|E(G)|$:

Base case:

Suppose $|E(G)| = 0$. Since G is a connected graph with no edges, $G = (\{v\}, \emptyset) = K_1$. Let $T = G$.

Induction hypothesis: Suppose that if G is a simple connected graph with $|E(G)| = k$, then G contains a subgraph T where T is a tree and $V(T) = V(G)$.

Let G be a simple connected graph with $|E(G)| = k + 1$.

Claim: G contains a subgraph T where T is a tree and $V(T) = V(G)$.

Case 1: If G is a tree, then let $T = G$. Then $V(T) = V(G)$

Case 2: G is not a tree. Thus G contains a cycle. Let e be an edge in this cycle. Then e is not a cut-edge (bridge).

Thus $G' = G - e$ is a simple connected graph with k edges. Thus by the induction hypothesis, G' contains a subgraph T where T is a tree and $V(T) = V(G')$.

Since $G' = G - e$, $V(G) = V(G') = V(T)$. Thus T is a subgraph of G where T is a tree and $V(T) = V(G)$. Thus the claim holds.

Alternative induction proof.

Proof by induction on $|E(G)|$:

Base case:

Suppose $|E(G)| = 0$. Since G is a connected graph with no edges, $G = (\{v\}, \emptyset) = K_1$. Let $T = G$.

Induction hypothesis: Suppose that if G is a simple connected graph with $|E(G)| < k + 1$, then G contains a subgraph T where T is a tree and $V(T) = V(G)$.

Let G be a simple connected graph with $|E(G)| = k + 1$.

Claim: G contains a subgraph T where T is a tree and $V(T) = V(G)$.

Let $e \in E(G)$ and let $G' = G - e$. Note $G' = G - e$ is a simple graph with k edges.

Case 1: G' is connected.

Thus $G' = G - e$ is a simple connected graph with k edges. Thus by the induction hypothesis, G' contains a subgraph T where T is a tree and $V(T) = V(G')$.

Since $G' = G - e$, $V(G) = V(G') = V(T)$. Thus T is a subgraph of G where T is a tree and $V(T) = V(G)$. Thus the claim holds.

Case 2: G' is not connected. Then G' has 2 connected components, G_1 and G_2.

$|E(G_i)| < E(G') = k$. Thus by the induction hypothesis, G_i contains a subgraph T_i where T_i is a tree and $V(T_i) = V(G_i)$.

Let $T = (V(T_1) \cup V(T_2), E(T_1) \cup E(T_2) \cup \{e\})$ – i.e, create T from T_1 and T_2 by adding in the edge e.

Note $V(T) = V(T_1) \cup V(T_2) = V(G') = V(G)$. Note T is connected since T_1 and T_2 are connected and all vertices in T_1 are reachable from all vertices in T_2 via a path that goes through e. T is also a tree since T_1 and T_2 are trees and e is a bridge. Thus T is a subgraph of G where T is a tree and $V(T) = V(G)$. Thus the claim holds.

7.) Define: G is $k-$vertex colorable if $V(G) = V_1 \cup V_2 \cup ... \cup V_k$ where $V_i \cap V_j = \emptyset \; \forall i \neq j$ and if 2 vertices $v, v' \in V_\ell$ for some $\ell \in \{1, ..., k\}$, then v and v' are not adjacent.

A graph H is $k-$edge colorable if $E(H) = E_1 \cup E_2 \cup ... \cup E_k$ where $E_i \cap E_j = \emptyset \; \forall i \neq j$ and if 2 edges $e, e' \in E_\ell$ for some $\ell \in \{1, ..., k\}$, then e and e' are not incident to the same vertex.

Let G be a simple connected graph. Show that G is $k-$edge colorable if and only if its line graph $L(G)$ is $k-$vertex colorable.

Proof: Let $L = L(G)$. If $E(G) = \{e_1, ..., e_m\}$, then $V(L) = \{e'_1, ..., e'_m\}$, where e'_p is the vertex in L corresponding to the edge e_p in G. Also, $< e'_i, e'_j > \in E(L)$ if and only if e_i, e_j are incident to the same vertex in G.

($\Rightarrow$) Suppose G is $k-$edge colorable. Then $E(G) = E_1 \cup E_2 \cup ... \cup E_k$ where $E_i \cap E_j = \emptyset$ $\forall i \neq j$ and if 2 edges $d, e \in E_\ell$ for some $\ell \in \{1, ..., k\}$, then d and e are not incident to the same vertex.

Let $E'_i = \{e'_j \in V(L) \mid e_j \in E_i\}$. Thus $V(L) = E'_1 \cup E'_2 \cup ... \cup E'_k$ where $E'_i \cap E'_j = \emptyset$ $\forall i \neq j$.

Suppose in L, the vertices $e'_i, e'_j \in E'_\ell$ for some $\ell \in \{1, ..., k\}$. Then their corresponding edges in G, e_i, e_j are in E_ℓ and thus are not incident to the same vertex in G. Thus $< e'_i, e'_j > \notin E(L)$. Thus in L, the vertices e'_i, e'_j are not adjacent and $E'_1 \cup E'_2 \cup ... \cup E'_k$ is a proper k-coloring of L. Hence L is $k-$vertex colorable.

($\Leftarrow$) Suppose L is $k-$vertex colorable. Then $V(L) = E'_1 \cup E'_2 \cup ... \cup E'_k$ where $E'_i \cap E'_j = \emptyset$ $\forall i \neq j$ and if 2 vertices $v, w \in E_\ell$ for some $\ell \in \{1, ..., k\}$, then v and w are not adjacent.

Let $E_i = \{e_j \in E(G) \mid e'_j \in E'_i\}$. Thus $E(G) = E_1 \cup E_2 \cup ... \cup E_k$ where $E_i \cap E_j = \emptyset$ $\forall i \neq j$.

Suppose in G, the edges $e_i, e_j \in E_\ell$ for some $\ell \in \{1, ..., k\}$. Then their corresponding vertices in L, e'_i and e'_j are in E'_ℓ and thus are not adjacent in L. Thus $e_i, e_j \in E(G)$ are not incident to the same vertex in G. . Hence G is $k-$edge colorable.

Note: One can instead use e_i to represent edges in G as well as vertices in L, but you also must be careful to be clear when e_i is an edge and when e_i is a vertex.

Alternative proof: Let $L = L(G)$. Then $E(G) = \{e_1, ..., e_m\} = V(L)$, Also, $< e_i, e_j > \in E(L)$ if and only if the edges e_i, e_j are incident to the same vertex in G. Thus $< e_i, e_j > \notin E(L)$ if and only if e_i, e_j are not incident to the same vertex in G.

G is $k-$edge colorable $\Leftrightarrow$ $E(G) = E_1 \cup E_2 \cup ... \cup E_k$ where $E_i \cap E_j = \emptyset \; \forall i \neq j$ and if 2 edges $d, e \in E_\ell$ for some $\ell \in \{1, ..., k\}$, then d and e are not incident to the same vertex in G. $\Leftrightarrow$ $V(L) = E_1 \cup E_2 \cup ... \cup E_k$ where $E_i \cap E_j = \emptyset \; \forall i \neq j$ and if 2 vertices $d, e \in E_\ell$ for some $\ell \in \{1, ..., k\}$, then $< d, e > \notin E(L)$ $\Leftrightarrow$ L is $k-$vertex colorable.