Defn: $F$ is a *forest* if $F$ does not contain any cycles.

Defn: $T$ is a *tree* if it is a connected forest (ie, a connected graph with no cycles).

Thm 5.2: For any tree $T$, $V(T) = E(T) + 1$.

Thm 5.6: The following are equivalent:

i.) $G$ is a tree.

ii.) $G$ is acyclic and connected (by defn).

iii.) $G$ is acyclic and $|V(G)| = |E(G)| + 1$.

iv.) $G$ is connected and $|V(G)| = |E(G)| + 1$.

v.) $G$ is connected and every edge is a cut edge.

vi.) $\exists$ exactly one path between every pair of vertices in $G$.

vi.) $G$ is acyclic and adding an edge creates a unique cycle.

Defn: $T$ is a *spanning tree* for a connected graph $G$ if

i.) $T$ is a tree,
ii.) $T$ is a subgraph of $G$, and
iii.) $V(T) = V(G)$.

Lemma: $G$ is connected iff $G$ has a spanning tree.

Cor 5.1: If $G$ is connected, then $V(G) \leq E(G) + 1$.

Algorithms for finding a (rooted) spanning tree in a connected graph:

i.) Breadth First Search (BFS)
ii.) Depth First Search (DFS)

} not greedy

Algorithms for finding minimal spanning tree in a connected graph:

non-rooted

i.) Kruskal's
ii.) Prim's

Algorithms for finding shortest paths and/or rooted spanning tree(s) in a connected graph (ex: sink tree):

i.) Dijkstra's ← BFS (not minimal spanning tree)
ii.) Bellman Ford ← not greedy

Theorems ⟹ greedy gives optimal in these cases

**Thm 5.7:** Let $T_{kr}$ = spanning tree of $G$ constructed via Kruskal's algorithm. Then $T_{kr}$ is a minimum spanning tree for $G$.

Pf: Let $E(T_{kr}) = \{e_1, ..., e_{|V(T_{kr})|-1}\}$
where $w(e_j) \leq w(e_{j+1})$ $\forall j = 1, ..., |V(T_{kr})| - 2$

Let $T$ be a spanning tree for $G$ such that $T \neq T_{kr}$.
Then $\exists j$ such that $e_j \notin T$.

Let $i(T) = min\{i \mid e_i \notin E(T)\}$

Let $i(T_{kr}) = |V(T_{kr})|$

Note that $i(T) < |V(T_{kr})|$ if $T \neq T_{kr}$

Let $m = \boxed{max}\{i(T) \mid T$ is a minimum spanning tree for $G\}$

Tm to make similar to Tkr ? (= Tkr ?)

Let $T_m$ be a minimum spanning tree such that $i(T) = m$. $\{e_1, ..., e_{m-1}\} \subset E(T_m)$

Case 1

If $m = |V(T_{kr})|$, then $T_m = T_{kr}$, and thus $T_{kr}$ is a minimum spanning tree for $G$. ← For this case, done

Case 2

If $m < |V(T_{kr})|$, then $e_m \notin T_m$. Thus $T_m \cup e_m$ contains a cycle $C$. ← come up w/ contradiction

$C \not\subset T_{kr}$ since $T_{kr}$ is a tree.

Thus $\exists\, \widehat{e} \in C$ such that $\widehat{e} \notin T_{kr}$

Since $C \subset T_m \cup e_m$ and $\widehat{e} \neq e_m$, $\widehat{e} \in T_m$.

Let $\widehat{T} = T_m - \widehat{e} + e_m$.

**Claim 1:** $\widehat{T}$ is a tree.
Note $|E(\widehat{T})| = |E(T_m)| = |V(T_m)| - 1 = |V(\widehat{T})| - 1$
$\widehat{T}$ is connected. Thus $\widehat{T}$ is a tree.

**Claim 2:** $\widehat{T}$ is a minimum spanning tree.

Note $w(\widehat{T}) = w(T_m) - w(\widehat{e}) + w(e_m)$.

$\widehat{e}$ is in this set $\Rightarrow w(e_m) \leq w(\widehat{e})$

**subclaim:** $w(\widehat{e}) \geq w(e_m)$.
$w(e_m) = min\{w(e) \mid e \in E(G) - \{e_1, ..., e_{m-1}\}$ and $\{e_1, ..., e_{m-1}, e\}$ is acyclic$\}$

$\widehat{e} \notin E(T_{kr}) \Rightarrow \widehat{e} \neq e_j$ for $j = 1, ..., m-1$.

$\{e_1, ..., e_{m-1}, \widehat{e}\} \subset T_m \Rightarrow \{e_1, ..., e_{m-1}, \widehat{e}\}$ is acyclic. ∎

Thus $w(e_m) \leq w(\widehat{e})$. Thus the subclaim is true and $w(\widehat{T}) \leq w(T_m)$

Thus $\widehat{T}$ is a minimum spanning tree.
$\{e_1, ..., e_m\} \in \widehat{T}$. Thus $i(\widehat{T}) > m$, a contradiction.

$\Rightarrow m \neq |V(T_m)|$

$\Rightarrow T_{kr}$ is a min sp tree