All graphs are finite in this class unless otherwise stated, i.e. $|V(G)|$ finite

Lemma: If T is a tree with $|V(T)| = n > 1$, then $\exists u \in V(T)$ such that $\delta(u) = 1$.

Since T is a tree, T is connected. Thus $|V(T)| > 1$ implies $\delta(v) \geq 1 \ \forall v \in V(T)$.

Proof by contradiction: Suppose $\nexists u \in V(T)$ such that $\delta(u) = 1$.

extra hypoth.

Then $\forall v \in V(T)$, $\delta(v) > 1$.

Let $v_1, v_2, \ldots, v_k$ be a longest path in T (since $V(T)$ is finite, a longest path exists).

$\delta(v_k) > 1$ implies $\exists u \in V(T)$ such that $w \neq v_{k-1}$.

check this is a path

If $w = v_i$ for $i < k-1$, then $v_i, v_{i+1}, \ldots, v_k, w$ is a cycle, contradicting that T is a tree. Thus $v_1, v_2, \ldots, v_k, w$ is a path in T

But this path is longer than $v_1, v_2, \ldots, v_k$, contradicting that we took a longest path in T.

Thus we have a contradiction and hence $\exists u \in V(T)$ such that $\delta(u) = 1$.

Note: we could have modified the above proof to show that $\exists$ 2 vertices in $V(T)$ with degree one.

Lemma 2.1: If T is a tree, then $|E(T)| = |V(T)| - 1$.

Proof by induction on $n = |V(T)|$.

Suppose $n = 1$. ← base case

Then $V(T) = \{v\}$ and $E(T) = \emptyset$.

Thus $|V(T)| - 1 = 1 - 1 = 0 = |E(T)|$. ✓

Induction hypothesis: If T is a tree with $n-1$ vertices, then $|E(T)| = |V(T)| - 1$.

Claim: If T is a tree with n vertices, then $|E(T)| = |V(T)| - 1$.

Let T be a tree with n vertices. Let $u \in V(T)$ such that $\delta(u) = 1$.

↑ hypothesis of claim

Let $< u, w > \in E(T)$.

Let $T' = (V(T) - \{u\},\ E(T) - \{< u, w >\})$ $|V(T')| = n-1$

Claim: T' is a tree:

If T' contains a cycle, then T contains a cycle. ⇒ Thus T' does not contain a cycle

If $x, y \in V(T') \subset V(T)$, then $\exists$ an $x - y$ path in T. Note this path does not contain the vertex u nor the edge $< u, w >$ since $\delta(u) = 1$. Thus this $x - y$ path lives in T' and T' is connected.

Thus T' is a tree. Note $|V(T')| = n - 1$.

By the induction hypothesis, $|E(T')| = |V(T')| - 1$.

Thus $|E(T)| - 1 = (|V(T)| - 1) - 1$.

Therefore $|E(T)| = |V(T)| - 1$ $\square$

Lemma: If T is a tree and $e_0 \in E(T)$, then $T - e_0 = T_1 \cup T_2$ where $T_1 \cup T_2 = \emptyset$ and T_1 and T_2 are trees.

Pf: Let $e_0 =< u, w >$. If $T - e_0$ is connected, then there exists a path $w, v_1, \ldots, v_k, u$ in $T - e_0$. But then $w, v_1, \ldots, v_k, u, w$ is a circuit in T. But T is a tree. Thus $T - e_0$ is not connected. Hence e_0 is a cut edge. Recall removing a minimal edge cut disconnects a graph into two connected components. Thus $T - e_0 = T_1 \cup T_2$ where T_i are connected. If T_1 or T_2 contains a cycle, then so does T. Hence T_1 and T_2 are trees.

Proof 2:

Lemma 2.1: If T is a tree, then $|E(T)| = |V(T)| - 1$.

Proof by induction on $m = |E(T)|$.

Suppose $m = 0$

Then $V(T) = \{v\}$ and $E(T) = \emptyset$.

Thus $|V(T)| - 1 = 1 - 1 = 0 = |E(T)|$.

Induction hypothesis: If T is a tree with $|E(T)| < m$, then $|E(T)| = |V(T)| - 1$.

Claim: If T is a tree with $m > 0$ edges, then $|E(T)| = |V(T)| - 1$.

Let T be a tree with $m > 0$ edges. Take $e_0 \in E(T)$.

Then $T - e_0 = T_1 \cup T_2$ where T_1 and T_2 are trees.

Since $E(T_i) \subset E(T) - \{e_0\}$, $|E(T_i)| < m$.

By the induction hypothesis, $|E(T_i)| = |V(T_i)| - 1$

$E(T) = E(T_1) \cup E(T_2) \cup \{e_0\}$ and $E(T_1) \cap E(T_2) = \emptyset$. Thus $|E(T)| = |E(T_1)| + |E(T_2)| + 1$.

$V(T) = V(T_1) \cup V(T_2)$ and $V(T_1) \cap V(T_2) = \emptyset$. Thus $|V(T)| = |V(T_1)| + |V(T_2)|$.

Hence $|E(T)| = |E(T_1)| + |E(T_2)| + 1$
$= |V(T_1)| - 1 + |V(T_2)| - 1 + 1 = |V(T)| - 1.$