Let G be a connected graph.

If $G = K_n$ = , then $\kappa(G) = n-1$

In all other cases:

$\kappa(G)$ = the size of a minimal vertex cut

= the minimum number of vertices one can remove such that the resulting subgraph is disconnected.

See Wed's notes

Thm 2.4: $\kappa(G) \leq \lambda(G) \leq min\{\delta(v) \mid v \in V(G)\}$

Case 1: Suppose $G = K_n$, then $\kappa(G) = n-1 = \lambda(G)$

Suppose $G \neq K_n$

Case 2: Let G be a graph such that $\lambda(G) = k$

Let $E^* = \{e_1, e_2, ..., e_k\}$ be a minimal edge cut of G.

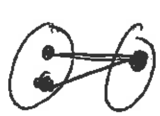
Claim: $G - E^* = G_1 \cup G_2$ where G_i, $i = 1, 2$ are the connected components of $G - E^*$.

Claim: $e_i =< u_i, v_i >$ where
$u_i \in V(G_1)$ and $v_i \in V(G_2)$ for $i = 1, ..., k$.

Let $U^* = \{u_1, ..., u_k\} \subset V(G_1)$. Note $|U^*| \leq k$.

Let $V^* = \underbrace{\{v_1, ..., v_k\}}_{\text{not this}} \subset V(G_2)$. Note $|V^*| \leq k$.

not this may contain repeats



subscript based on the e_i's (u_i is endpt of e_i; v_i " ")

$V(G_1) \neq U^*$

Case 2a: $\exists u \in V(G_1)$ such that $u \notin \{u_1, ..., u_k\}$.

Claim: u is not connected to v_1 in $G - U^*$.

Thus $G - U^*$ is disconnected and hence U^* is a vertex cut for G. Therefore $\kappa(G) \leq k = \lambda(G)$.

Case 2b: $\exists v \in V(G_2)$ such that $v \notin \{v_1, ..., v_k\}$.

Claim: v is not connected to u_1 in $G - V^*$.

Thus $G - V^*$ is disconnected and hence V^* is a vertex cut for G. Therefore $\kappa(G) \leq k = \lambda(G)$.

proof is similar to case 2a

Case 2c: $V(G_1) = U^* = \{u_1, ..., u_k\}$ and $V(G_2) = V^* = \{v_1, ..., v_k\}$. note vertices in U^* need not be distinct. Similarly for V^*

Since G is not a complete graph,
$\exists x, y \in V(G) = \{u_1, ..., u_k, v_1, ..., v_k\}$
such that $< x, y > \notin E(G)$.

WLOG assume $x = u_1$.

Let $N(u_1) = \{u_{i_1}, ..., u_{i_\ell}, v_{j_1}, ..., v_{j_m}\}$
where $u_{i_s} \in U^*$ and $v_{j_t} \in V^*$ $\forall s, t$.
"$V(G_1)$" "$V(G_2)$"

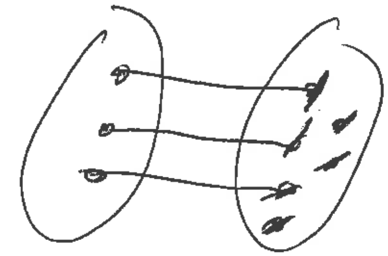
Note $x = u_1$ is not connected to y in $G - N(u_1)$.
Thus $N(u_1)$ is a vertex cut for G.

Claim $|N(u_1)| = \delta(u_1) \leq k$.

Define $f : N(u_1) \to E^*$ by

$f(v_{j_t}) =< u_1, v_{j_t} >$ and

$f(u_{i_s}) =< u_{i_s}, v_p >$
where $p = min\{j \mid v_j$ is adjacent to $u_{i_s}\}$ OK



$u_{i_s} \in G_1 \Rightarrow \langle u_{i_s}, v_p \rangle \in E^*$
$v_p \in G_2$

Note f is a well-defined 1:1 function.

Thus $|N(u_1)| \leq |E^*|$.

$G = (V, E)$ is *k-connected* if removing any set of $k-1$ vertices in G does not disconnect it and $G \neq K_n$

K_n is $n-1$-connected.

k connected implies $k-1$ connected if $k > 1$

connected = 1-connected except $K_1 = \bullet$ is also a connected graph

$\kappa(G) =$ connectivity of $G = max\{k \mid G\, k\text{−}connected\}$ $\blacksquare$