This gives us an algorithm for determining if sequence is graphic

Theorem 2.2 (Havel-Hakimi): Consider a list $s = [d_1, d_2, ..., d_n]$ of n numbers in descending (non-increasing) order. This list is graphic if and only if $s^* = [d_1^*, d_2^*, ..., d_{n-1}^*]$ of $n-1$ numbers is graphic as well, where

$$d_i^* = \begin{cases} d_{i+1} - 1 & \text{for i} = 1, 2, \ldots, d_1 \\ d_{i+1} & \text{otherwise} \end{cases}$$

→ I.e., s is graphic iff s^* is graphic.

Proof: Let $s = [d_1, d_2, ..., d_n]$ where $d_i \geq d_{i+1}$, $d_i \in \{0, 1, 2, ...\}$

Let $s^* = [d_1^*, d_2^*, ..., d_{n-1}^*]$ where

$$d_i^* = \begin{cases} d_{i+1} - 1 & \text{for i} = 1, 2, \ldots, d_1 \\ d_{i+1} & \text{otherwise} \end{cases}$$

Thus $d_i = \begin{cases} d_1 & \text{i} = 1 \\ \boxed{d_{i-1}^*} + 1 & \text{for i} = 2, 3, \ldots, d_1 + 1 \\ d_{i-1}^* & \text{otherwise} \end{cases}$

Scratchwork: Added vertex u and $\delta(u) = d_1$

Find $G=(V,E)$ simple graph

($\Leftarrow$) Suppose s^* is graphic.

Claim s is graphic. ← Goal: Find G w/ degree seq s

s^* graphic implies $\exists$ simple graph G^* with degree sequence s^*.

Let $G^* = (V^*, E^*)$ where $V^* = \{v_1, ..., v_{n-1}\}$ and $\delta(v_i) = d_i^*$.

Let $V = \{u, v_1, ..., v_{n-1}\}$

Let $E = E^* \cup \{< u, v_i > \mid i = 1, ..., d_1\}$

$$\text{Then } \delta(w) = \begin{cases} d_1 & w = u \\ d_i^* + 1 & w = v_i,\ i = 1, ..., d_1 \\ d_i^* & \text{else} \end{cases}.$$

Thus $G = (V, E)$ has degree sequence s. Note G is a simple graph. Thus s is graphic.

We will create E by adding the following edges to E^*:

d_1 edges so that $\delta(u) = d_1$: $\begin{cases} \langle u, v_1 \rangle \\ \langle u, v_2 \rangle \\ \vdots \\ \langle u, v_{d_1} \rangle \end{cases}$