Study guide for the first exam

Math 2374, Fall 2006

1. Basic vector material (Chapter 1)
   - (a) Comments: the initial sections of this course are background material for the rest of the course. The following may help you organize your studying of the diverse topics.
   - (b) Key items for exam 1
     - i. Computing 2×2 and 3×3 determinants: although these become more important later in the course, you can use them now to help you memorize the cross product.
     - ii. Dot products and cross products: these are used all the time, throughout the whole course. Make sure you understand and can compute them.
     - iii. Parametrizations of lines and equations of planes: these form an important basis of the course. A good understanding of them will be important.
     - iv. Vectors in $\mathbf{R}^n$. Be able to find magnitudes of vectors.
     - v. Matrices. Multiply matrices times vectors, matrices times matrices.
   - (c) Notes
     - i. We don't cover cylindrical and spherical coordinates (Section 1.4) until later in the course.
   - (d) Sample book problems: 1.3 #15(d), #16(b), #26, #30, 1.5 #8

2. Functions and graphing (Section 2.1)
   - (a) Three-dimensional graphing: the only graphs in three dimensions we might ask you to sketch are quadric surfaces, planes, cylinders, lines, as well as portions or combinations of these.
   - (b) Level sets: level curves for functions of two variables and level surfaces for functions of three variables. Be able to sketch a few level curves as in the homework.
   - (c) Sample book problems: 2.1 #1(a), #2(b), #5

3. Derivatives (a big focus of the exam)
   - (a) Partial derivatives (Section 2.3)
     - i. Key items: understand and compute partial derivatives
     - ii. Methods: limit definition, one-variable calculus techniques.
     - iii. Sample book problems: 2.3 #2(b), #3(b)

(b) The derivative (Section 2.3)

i. Key idea 1: the derivative is represented by the matrix of partial derivatives

ii. Key idea 2: use the derivative to write a linear approximation of a function f near a point $\mathbf{a}$.

iii. Key idea 3: a function being differentiable at a point means it is nearly linear around that point.

iv. Key idea 4: if the partial derivatives in the matrix of partial derivatives are continuous at a point, then the function is differentiable.

v. Note that for $f : \mathbf{R}^2 \to \mathbf{R}$, the linear approximation is the tangent plane.

vi. For a scalar-valued function, the derivative can be written as vector (the gradient).

vii. Sample book problems: 2.3 #6(b). #12(b), #13(c), #14(c)

(c) Introduction to paths and curves (Section 2.4)

i. Kknow that a curve can be parametrized by a function $\mathbf{c}(t)$, that $\mathbf{c}'(t)$ is the velocity of an object with position $\mathbf{c}(t)$, and $\mathbf{c}'(t)$ is tangent to the path.

ii. Be able to compute a tangent line to a curve.

iii. Sample book problems: 2.4 #15, #17

(d) The chain rule (Section 2.5)

i. Key idea: The chain rule gives the derivative of a composition of functions.

ii. Key formula: $D(f \circ g)(\mathbf{a}) = Df(g(\mathbf{a}))Dg(\mathbf{a})$

iii. Note: Formulas for partial derivatives can be derived from above formula, but be careful to evaluate partial derivatives of f at the point $g(\mathbf{a})$.

iv. Sample book problems: 2.5 #2(f), #5(b), #9, #13

4. The gradient and the directional derivative (Section 2.6)
(we assume functions are differentiable)

(a) The gradient

i. Key idea: for scalar-valued function f, the gradient ∇f is like the matrix of partial derivatives Df, except that the gradient is a vector rather than a matrix.

ii. The gradient is a vector whose magnitude and direction have physical meaning.

A. The gradient points in the direction where f increases most rapidly.

B. The magnitude of the gradient indicates the rate of change in f in that direction.

iii. Since the gradient is perpendicular to level sets of f, you can use the gradient to find tangent tangent planes to surfaces.

iv. Sample book problems: 2.6 #4(c), #7(c)

(b) The directional derivative

i. Key idea: the directional derivative is a generalization of the partial derivative. The directional derivative $D_{\mathbf{u}}f$ gives the rate of change of f in the direction specified by $\mathbf{u}$ ($D_{\mathbf{u}}f$ represents slope in that direction).

ii. Important formula: $D_{\mathbf{u}}f(\mathbf{a}) = \nabla f(\mathbf{a})\cdot\mathbf{u}$ (alternatively $D_{\mathbf{u}}f(\mathbf{a}) = \|\nabla f(\mathbf{a})\| \cos\theta$)

iii. Don't forget: $\mathbf{u}$ must be a **unit vector**.

iv. Although the gradient is a vector, the directional derivative is a scalar.

v. If $\mathbf{u}$ is perpendicular to the gradient, then $D_{\mathbf{u}}f = 0$. If $\mathbf{u}$ points in the same direction as the gradient, then $D_{\mathbf{u}}f = \|\nabla f\|$. If $\mathbf{u}$ points in the opposite direction of the gradient, then $D_{\mathbf{u}}f = -\|\nabla f\|$.

vi. Sample book problems: 2.6 #3(b), #20

Study guide for the second exam

Math 2374, Fall 2006

1. Higher order partial derivative (section 3.1)

   (a) Be able to compute all secord-order partial derivatives

   (b) Clairaut's Theorem: mixed partials are equal for twice continuously differentiable functions

   (c) Sample book problems: 3.1 #2, #15(a)

2. Parametrized curves, length, and vector fields (Chapter 4)

   (a) Paths (parametrized curves)

      i. Key idea: A vector-valued function of one variable (e.g., $\mathbf{c}(t)$) parametrizes a path.

      ii. Find parametrizations of curves such as lines, circles, ellipses, and segments of these (needed especially to compute path and line integrals over curves)

      iii. A parametrization needs both a function $\mathbf{c}(t)$ and a range $a \leq t \leq b$.

      iv. Can parametrize in two directions (orientations). (Could think of unit tangent vector $\mathbf{T} = \mathbf{c}'(t)/\|\mathbf{c}'(t)\|$ as specifying direction.)

   (b) Path length

      i. Key idea: path length element of $\mathbf{c}(t)$ is $ds = \|\mathbf{c}'(t)\| dt$.

      ii. The length of a curve C parametrized by $\mathbf{c}(t)$ for $a \leq t \leq b$ is $L(C) = \int_C ds = \int_a^b \|\mathbf{c}'(t)\| \, dt$.

      iii. Can parametrize a curve in multiple ways, but path length is independent of parametrization.

   (c) Vector fields

      i. For this class, our main use of vector fields is when we compute line integrals (and later surface integrals) of vector-valued functions (vector fields).

      ii. It's good to know how to sketch vector fields. In particular, it will allow you to estimate values of line integrals to double-check your answers.

   (d) Divergence and curl

      i. Key idea for divergence: measures outflow per unit volume of fluid flow

      ii. Key idea for curl: measures rotation of fluid flow

      iii. $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$

      iv. $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$

   (e) Sample book problems: 4.2 #6, #9, 4.3 #5, 4.4 #11, #14

3. Double integrals (sections 5.1 – 5.4)

   (a) Key idea: although defined by Riemann sums over rectangles, these integrals can be computed through iterated integrals.
   (b) Be able to compute bounds for iterated integrals, especially for the different orders of integration.
   (c) Remember: outer limits must be constant; inner limits can depend only on variables from the outside integral.
   (d) Finding limits and changing order of integration are easiest if you draw pictures. Inequalities are helpful, too.
   (e) Sample book problems: 5.1 #8, 5.2 #2(b), #7, 5.3 #2(e), #4, 5.4 #2(c), 10, 13

4. Triple integrals (section 5.5)

   (a) Key idea: although defined by Riemann sums over boxes, these integrals can be computed through iterated integrals.
   (b) One trick: computing bounds for iterated integrals, especially for the different orders of integration.
   (c) Remember: outer limits must be constant; inner limits can depend only on variables from the outside integral(s).
   (d) Finding limits and changing order of integration are easiest if you draw pictures. Inequalities are helpful, too.
   (e) Sample book problems: 5.5 #6, #9, #21, #22

5. Path integrals of scalar functions (Section 7.1)

   (a) Key idea: Integrate scalar function $f(\mathbf{x})$ along curve (i.e., $f(\mathbf{c}(t))$) using the ds from path length.
   (b) Formula: $\int_C f\,ds = \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|dt$
   (c) If $f(\mathbf{c})$ is density of wire, then $\int_C f\,ds$ is mass of wire.
   (d) If $f(\mathbf{c}) = 1$, then $\int_C f\,ds = \int_C ds$ is length of C.
   (e) $\int_C f\,ds$ is independent of parametrization of C.
   (f) Sample book problems: 7.1 #3(b), #7(a), #10

6. Line integrals of vector-valued functions (Section 7.2)

   (a) Key idea: Integrate tangent component of $\mathbf{F}(\mathbf{x})$ along curve (i.e. $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}$) using above ds.
   (b) Formula: $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$.
   (c) If $\mathbf{F}$ is a force field, then $\int_C \mathbf{F} \cdot d\mathbf{s}$ is the work done by the force field on a particle moving along C.

(d) $\int_C \mathbf{F} \cdot d\mathbf{s}$ is independent of parametrization of C, but depends on the direction of C, as $\int_{C^-} \mathbf{F} \cdot d\mathbf{s} = -\int_C \mathbf{F} \cdot d\mathbf{s}$

(e) Sample book problems: 7.2 #2(c), #7, #14

7. Green's Theorem (section 8.1)

(a) Key idea: If computing a line integral of a vector field $\mathbf{F}$ over a closed curve in 2D, you can convert it to a double integral (if $\mathbf{F}$ is defined in the whole interior of the curve).

(b) Formula: $\displaystyle\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$

(c) Sometimes, we write $\mathbf{F} = (P, Q)$, in which case Green's theorem is written $\displaystyle\int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$

(d) Important: you need a "positively oriented" boundary $C = \partial D$ correctly. The region D must be on your left as you move along C. (This means inner boundaries will go the opposite direction of outer boundaries.)

(e) Other application: you can use Green's theorem to calculate the area of the region D, which is $\iint_D dA$, by letting, for example, $\mathbf{F} = \frac{1}{2}(-y, x)$ so that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$.

(f) Sample book Problems: 8.1 #2, #3(b), #5