Study guide for third exam

Math 2374, Fall 2006

1. Cylindrical and spherical coordinates (Section 11.8)
   (a) We'll use them for changing variables.
   (b) ~~Sample book problems: 1.4 #4, #12~~
2. Change of variables (Section 13.9)
   (a) In double integrals
      i. Key idea: Evaluate integral in new region over new coordinates with new area measure $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$.
      ii. Formula: $\iint_D f(x,y)dx\,dy = \iint_{D^*} f(\mathbf{T}(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$.
      iii. Important special case: polar coordinates, where $dx\,dy = r\,dr\,d\theta$.
   (b) In triple integrals
      i. Key idea: Evaluate integral in new region over new coordinates with new volume measure $dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$.
      ii. Formula: $\iiint_W f(x,y,z)dx\,dy\,dz = \iiint_{W^*} f(\mathbf{T}(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$.
      iii. Important special cases: cylindrical coordinates ($dx\,dy\,dz = r\,dr\,d\theta\,dz$), spherical coordinates ($dx\,dy\,dz = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$)
   ~~(c) Sample book problems: 6.1 #2, 6.2 #2, #14, #23, #26, #30~~
3. Parametrized surfaces ~~(Section 7.3)~~
   (a) Parametrize key surfaces: spheres, cylinders, cones, planes, surface of form $z = h(x,y)$.
   (b) Tangent vectors: $\mathbf{T}_u = \frac{\partial \mathbf{\Phi}}{\partial u}$ and $\mathbf{T}_v = \frac{\partial \mathbf{\Phi}}{\partial v}$
   (c) Normal vector: $\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v$
   (d) Use normal vector to find equation for tangent plane
   (e) Unit normal vector $\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|}$ specifies orientation. Positive side of surface is side with normal.
   ~~(f) Sample book problems: 7.3 #3, #7, #14~~
4. Surface area of a parametrized surface (Section 13.8)
   (a) Key idea: surface area element of $\mathbf{x} = \mathbf{\Phi}(u,v)$ is $dS = \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$
   (b) Formula: Surface area $\iint_D \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$.
   ~~(c) Sample book problems: 7.4 #5, #8, #13~~

5. Surface integrals of scalar-valued function (Section 14.5)

   (a) Key idea: Integrate $f(\mathbf{x})$ across surface (i.e., $f(\mathbf{\Phi}(u,v))$) using above dS.

   (b) Formula: $\iint_S f\, dS = \iint_D f(\mathbf{\Phi}(u,v))\, \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$.

   (c) Formula for graphs (special case of above formula):
   $\iint_S f\, dS = \iint_D f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dx\, dy$

   (d) ~~Sample book problems: 7.5 #3, #8, #14~~

6. Surface integrals of vector-valued functions (Section 14.5)

   (a) Key idea: Integrate normal component of $\mathbf{F}(\mathbf{x})$ across surface
   (i.e., $\mathbf{F}(\mathbf{\Phi}(u,v)) \cdot \mathbf{n}$) using above dS.

   (b) Formula: $\iint_S \mathbf{F} \cdot d\mathbf{S} = \pm \iint_D \mathbf{F}(\mathbf{\Phi}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$. (minus sign if $\mathbf{T}_u \times \mathbf{T}_v$ points in the opposite direction as $\mathbf{n}$)

   (c) Formula for graphs (special case of above formula, upward normal):
   $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(x,y,g(x,y)) \cdot \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) dx\, dy$

   (d) ~~Sample book problems: 7.6 #1, #6, #16~~

7. ~~Stokes' Theorem (Section 8.2)~~

   (a) ~~Key idea 1: to calculate circulation of $\mathbf{F}$ around closed curve C, you can choose any surface with boundary C and calculate flux integral of $\nabla \times \mathbf{F}$ over surface.~~

   (b) ~~Key idea 2: if calculating the surface integral of a vector field $\mathbf{G} = \nabla \times \mathbf{F}$ over a surface S, then you can either~~

      i. ~~convert it to the integral of $\mathbf{F}$ over the boundary ∂S, or~~

      ii. ~~change the surface S to any other surface S' with the same boundary $\partial S' = \partial S$ and compute the integral over S' rather than over S.~~

   (c) ~~Need positively oriented boundary: orient using the right hand rule (alternatively, walk on positive side of surface near boundary and surface is on left).~~

   (d) ~~Formula: $\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$~~

   (e) ~~Sample book problems: 8.2 #1, #9, #21, #25~~

8. Conservative vector fields (Section 14.3)

   We focused on conservative vector fields in R2
   will look at R3 case later.

   (a) Key idea: test if a vector field is conservative. If it is, your life got a lot easier (that is, if you're trying to compute a line integral of the vector field).

   (b) Fact: if a vector field $\mathbf{F}$ is conservative, then

      i. its line integral depends only on the endpoints (so is zero over closed curves)

      ii. $\mathbf{F} = \nabla f$ for some potential function f.

      iii. $\int_C \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{q}) - f(\mathbf{p})$, where $\mathbf{p}$ and $\mathbf{q}$ are the endpoints of the path.

   (c) Test for conservative vector fields:

i. In 2D where $\mathbf{F}$ is defined in all $\mathbf{R}^2$: $\mathbf{F}$ is path-independent if and only if $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$.

ii. ~~In 3D where $\mathbf{F}$ is defined in all $\mathbf{R}^3$ except for possibly a finite number of points: $\mathbf{F}$ is path-independent if and only if $\nabla \times \mathbf{F} = \mathbf{0}$.~~

(d) Don't forget the consequence of having a hole through the domain in $\mathbf{R}^2$.

(e) If $\mathbf{F}$ is conservative, find potential function f so that $\nabla f = \mathbf{F}$.

~~(f) Sample book Problems: 8.3 #2, #3, #9, #13(b)~~