14.2 Scalar Line Integrals:

Let $\mathbf{C} : [a, b] \to \mathbf{R}^n$ be a smooth curve. $f : \mathbf{R}^n \to R$, a scalar field.

$$\Delta s_k = \text{length of kth segment of path}$$
$$= \int_{t_{k-1}}^{t_k} ||\mathbf{C}'(t)||dt = ||\mathbf{C}'(t_k^*)||(t_k - t_{k-1}) = ||\mathbf{C}'(t_k^*)||\Delta t_k$$

for some $t_k^{**} \in [t_{k-1}, t_k]$

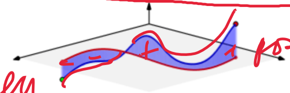
$$\int_{\mathbf{C}} f \; d\mathbf{r} \sim \Sigma_{i=1}^n f \; (\mathbf{C}(t_k^*))\Delta s_k = \Sigma_{i=1}^n f(\mathbf{C}(t_k^*))||\mathbf{C}'(t_k^{**})||\Delta t_k$$

$$\boxed{\text{Thus } \int_{\mathbf{C}} f \; ds = \int_a^b \underbrace{f \; (\mathbf{C}(t))}_{h}\underbrace{||\mathbf{C}'(t)||dt}_{w}}$$

$\int_{\mathbf{C}} f \; ds =$ area under curve $f(\mathbf{C})$

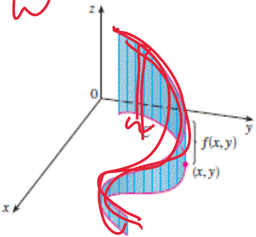
where area is positive above xy plane

and negative below xy plane



neg pos

https://en.wikipedia.org/wiki/Line_integral



https://brilliant.org/wiki/line-integral

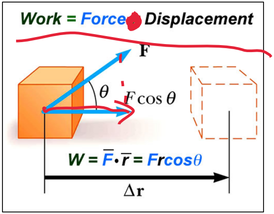
14.2 Vector Line integrals:

Let $\mathbf{C} : [a, b] \to \mathbf{R}^n$ be a smooth path. $F : \mathbf{R}^n \to R^n$, a vector field.

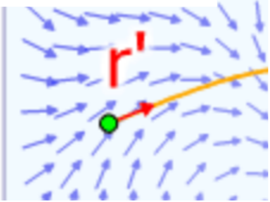
$$\mathbf{C}'(t_k^*) \sim \frac{\Delta \mathbf{C}_k}{\Delta t_k}$$

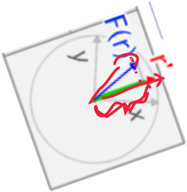
$$\int_{\mathbf{C}} F \cdot d\mathbf{r} \sim \Sigma_{i=1}^n F(\mathbf{C}(t_k^*)) \cdot \Delta \mathbf{C}_k = \Sigma_{i=1}^n F(\mathbf{C}(t_k^*)) \cdot \mathbf{C}'(t_k^*) \Delta t_k$$

$$\boxed{\text{Thus } \int_{\mathbf{C}} F \cdot d\mathbf{r} = \int_a^b F(\mathbf{C}(t)) \cdot \mathbf{C}'(t) dt}$$



F is a vector field.





http://physicssimplifiedforyou.blogspot.com/

https://en.wikipedia.org/wiki/Line_integral

Thm: Let $\mathbf{C} : [a, b] \to \mathbf{R}^n$ be a piecewise smooth path and let $\mathbf{D} : [c, d] \to \mathbf{R}^n$ be a reparametrization of $\mathbf{C}$. Then

Scalar line integral:

If $f : \mathbf{R}^n \to \mathbf{R}$ is continuous, then $\int_{\mathbf{D}} f \; ds = \int_{\mathbf{C}} f \; ds$

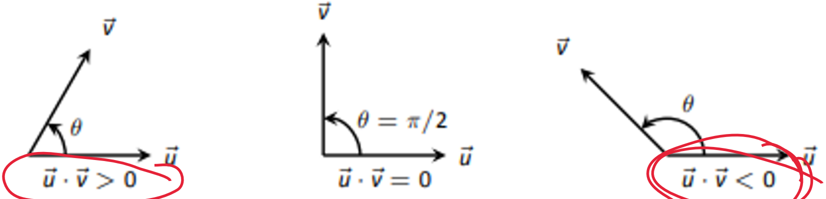
since area under curve does not depend on parametrization.

Vector line integral

If $F : \mathbf{R}^n \to \mathbf{R}^n$ is continuous, then

$$\int_{\mathbf{D}} F \cdot d\mathbf{r} = \int_{\mathbf{C}} F \cdot d\mathbf{r} \text{ if } \mathbf{D} \text{ is orientation-preserving.}$$

$$\int_{\mathbf{D}} F \cdot d\mathbf{r} = -\int_{\mathbf{C}} F \cdot d\mathbf{r} \text{ if } \mathbf{D} \text{ is orientation-reversing.}$$

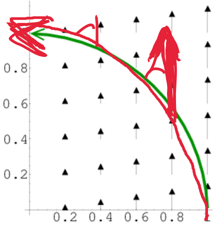


https://math.libretexts.org/Bookshelves/Calculus/Book:_Calculus_(Apex)/10:_Vectors/10.3:_The_Dot_Product

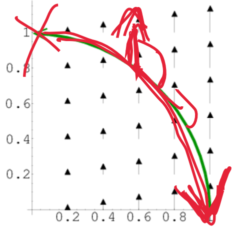
$$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_{C} \vec{F}\cdot d\vec{r} = \text{Work}$$

http://tutorial.math.lamar.edu/Classes/CalcIII/LineIntegralsVectorFields.aspx

positive



negat



Another notation (differential form): For simplicity, we will work in $\mathbf{R}^2$, but the following generalizes to any dimension.

Let $\mathbf{C}(t) = (x(t), y(t))$. Let $F(x, y) = (P(x, y), Q(x, y))$

where $x = x(t), y = y(t)$. Note $x'(t) = \frac{dx}{dt}$, $y'(t) = \frac{dy}{dt}$

Thus $dx = x'(t)dt$ and $dy = y'(t)dt$. Also, $d\mathbf{r} = (dx, dy)$

$$\int_{\mathbf{C}} F \cdot d\mathbf{r} = \int_{\mathbf{C}} (P(x, y), Q(x, y)) \cdot (dx, dy)$$

dy

dx

$$\int_{\mathbf{C}} F \cdot d\mathbf{r} = \int_{\mathbf{C}} P(x, y)dx + \int_{C} Q(x, y)dy$$

$$= \int_a^b P(x(t), y(t))\boxed{x'(t)dt} + \int_a^b Q(x(t), y(t))\boxed{y'(t)dt}$$

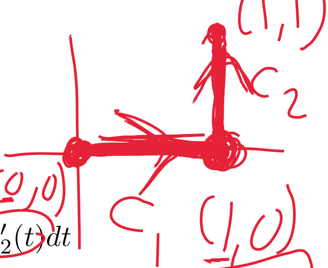
Work in x-dir

Work in y-direct

$C_2'(t) = (0,1)$

Let $F(x,y) = (x,y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

$C_2(t) = (1,t) \quad 0 \le t \le 1$



Notation 1: Work definition

$$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$$

$$= \int_0^1 F(\mathbf{C}_1(t)) \cdot \mathbf{C}_1'(t)dt + \int_0^1 F(\mathbf{C}_2(t)) \cdot \mathbf{C}_2'(t)dt$$

$$= \int_0^1 F(t,0) \cdot (1,0)dt + \int_0^1 F(1,t) \cdot (0,1)dt$$

$$= \int_0^1 (t,0) \cdot (1,0)dt + \int_0^1 (1,t) \cdot (0,1)dt$$

$$= \int_0^1 tdt + \int_0^1 tdt = 2(\tfrac{1}{2}t^2)|_0^1 = 1$$

$C_1(t) = (t,0)$

$0 \le t \le 1$

$C_1'(t) = (1,0)$

Let $F(x,y) = (x,y)$, let $C(t) = C_1(t) \cup C_2(t)$

$x \quad y$ $x \quad y$

$C_1(t) = (t,0), \;\; 0 \le t \le 1$ $C_2(t) = (1,t), \;\; 0 \le t \le 1$

$C_1'(t) = (1,0), \;\; 0 \le t \le 1$ $C_2'(t) = (0,1), \;\; 0 \le t \le 1$

$c_1'(t)dt = (1dt, 0dt)$

Thus along C_1, $dx = 1dt$ and $dy = 0dt$ $c_2' dt = (0dt, 1dt)$

and along C_2, $dx = 0dt$ and $dy = 1dt$

$dr = (dx, dy)$

Notation 2: differential form

$F = (x, y)$

$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$

$= \int_{C_1} F \cdot (dx, dy) + \int_{C_2} F \cdot (dx, dy)$

$= \int_{C_1} (x,y) \cdot (dx,dy) + \int_{C_2} (x,y) \cdot (dx, dy)$

plug in x

$= \int_{C_1} (xdx + ydy) + \int_{C_2} (xdx + ydy)$

$t \; dt \quad dt \quad t$

$= \int_0^1 (tdt + 0(0dt)) + \int_0^1 (1(0dt) + tdt) = 2(\frac{1}{2}t^2)|_0^1 = 1$

$wx \qquad wy$

c_2

c_1

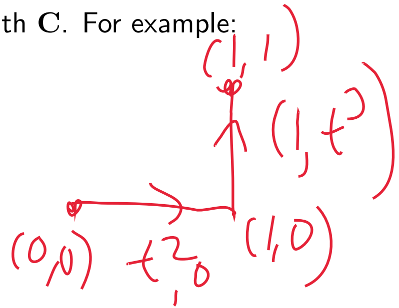
Note: Both of these methods are algebraically equivalent, so it doesn't matter which notation you use.

Let $F(x, y) = (x, y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Can use any parametrization for the path $\mathbf{C}$. For example:

$\mathbf{C}_1 : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}_1(t) = (t^2, 0)$

$\mathbf{C}_2 : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}_2(t) = (1, t^3)$



Notation 1: Work definition

$$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$$

$$= \int_0^1 F(\mathbf{C}_1(t)) \cdot \mathbf{C}_1'(t)dt + \int_0^1 F(\mathbf{C}_2(t)) \cdot \mathbf{C}_2'(t)dt$$

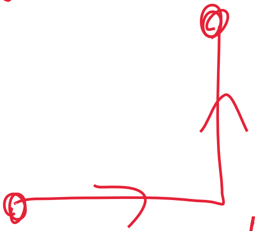
$$= \int_0^1 F(t^2, 0) \cdot (2t, 0)dt + \int_0^1 F(1, t^3) \cdot (0, 3t^2)dt$$

$$= \int_0^1 (t^2, 0) \cdot (2t, 0)dt + \int_0^1 (1, t^3) \cdot (0, 3t^2)dt$$

$$= \int_0^1 2t^3 dt + \int_0^1 3t^5 dt = \tfrac{1}{2}t^4|_0^1 + \tfrac{1}{2}t^6|_0^1 = 1$$

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Can use any parametrization for the path $\mathbf{C}$. For example:



$\mathbf{C}_1 : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}_1(t) = (t^2, 0)$

$\mathbf{C}_2 : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}_2(t) = (1, t^3)$

Notation 2: differential form

$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$

orientation preserving

$= \int_{C_1} F \cdot (dx, dy) + \int_{C_2} F \cdot (dx, dy)$

$= \int_{C_1} (x, y) \cdot (dx, dy) + \int_{C_2} (x, y) \cdot (dx, dy)$

$= \int_{C_1} [xdx + ydy] + \int_{C_2} [xdx + ydy]$

$= \int_0^1 t^2(2t)dt + \int_0^1 1(0) + t^3 3t^2 dt = \frac{1}{2}t^4|_0^1 + \frac{1}{2}t^6|_0^1 = 1$

Let $F(x,y) = (x,y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Methods for special cases:

Method: 14.3 Suppose $F = \nabla f$

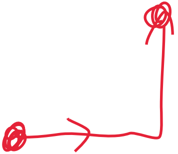
Claim F has path independent line integrals.

Wed

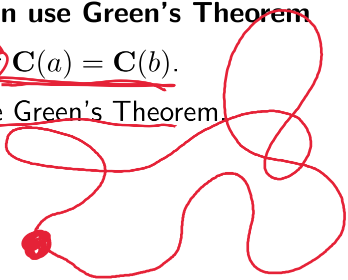
Method: 14.4 For closed curves, can use Green's Theorem

A path $\mathbf{C} : [a,b] \to \mathbf{R}^n$ is *closed* if $\mathbf{C}(a) = \mathbf{C}(b)$.

If curve is not closed, canNOT use Green's Theorem.



$C(a) = C(b)$



In Calculus I we had the **Fundamental Theorem of Calculus** that told us how to evaluate definite integrals. This told us,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Suppose that C is a **smooth** curve given by $\vec{r}(t)$, $a \le t \le b$. Also suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then,

$f(\vec{q}) - f(\vec{p})$

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$\vec{r}(b) = q$

$p = \vec{r}(a)$

def

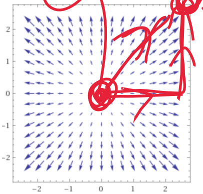
$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t))\right] dt \quad \text{by the chain rule}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \text{by FTC}$$

$\frac{\partial}{\partial x}\left(\frac{1}{2}x^2+k(y)\right) = \nabla f$

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).



Find f such that $\nabla f = F(x, y) = (x, y)$

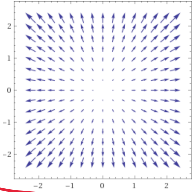
If $F(x, y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x}\,dx = \int x\,dx$

$f(x,y) = \frac{1}{2}x^2 + k(y)$

$\frac{\partial f}{\partial x}$ $\qquad$ $\frac{\partial f}{\partial y} = y$

Let $F(x,y) = (x,y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).



If $F(x,y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

$\boxed{\frac{\partial f(x,y)}{\partial x} = x}$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $\boxed{f(x,y) = \frac{x^2}{2} + k(y)}$

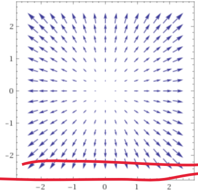
$\frac{\partial f}{\partial y} =$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2}{2} + k(y)\right) = 0 = k'(y) = y$$

$$\Rightarrow k(y) = \frac{1}{2}y^2 + C$$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ $\quad \frac{\partial f}{\partial y}$

Let $F(x, y) = (x, \boxed{y})$, let $C(t) =$ path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).



If $F(x, y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

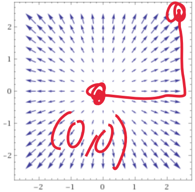
$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $\boxed{f(x, y) = \frac{x^2}{2} + \underline{k(y)}}$

$\frac{\partial f}{\partial y} = \frac{\partial(\frac{x^2}{2} + k(y))}{\partial y} = k'(y) = \boxed{y}$. Hence $k(y) = \frac{y^2}{2} +$ constant

Thus if we let $f(x, y) =$ $\frac{x^2}{2} + \frac{y^2}{2} + \cancel{C}$ 0

then $\nabla f =$ $\left\langle \frac{2x}{2} + 0, 0 + \frac{2y}{2} \right\rangle = \langle x, y \rangle$ ✓

Let $F(x, y) = (x, y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1). $(1,1)$



If $F(x, y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $f(x, y) = \frac{x^2}{2} + k(y)$

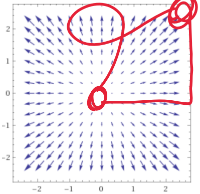
$\frac{\partial f}{\partial y} = \frac{\partial(\frac{x^2}{2} + k(y))}{\partial y} = k'(y) = y$. Hence $k(y) = \frac{y^2}{2}+$ constant

Thus if we let $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$, then $\nabla f = (x, y) = F$.

$$\boxed{\text{Hence } \int_C F \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

$f(1,1) - f(0,0)$

Let $F(x,y) = (x,y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).



If $F(x,y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $f(x,y) = \frac{x^2}{2} + k(y)$

$\frac{\partial f}{\partial y} = \frac{\partial(\frac{x^2}{2} + k(y))}{\partial y} = k'(y) = y$. Hence $k(y) = \frac{y^2}{2} +$ constant

Thus if we let $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$, then $\nabla f = (x,y) = F$.

Hence $\int_C F \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$$= f(1,1) - f(0,0) = \frac{1^2}{2} + \frac{1^2}{2} - 0 = 1$$

F is called *conservative* if $\boxed{F = \nabla f}$. In this case f is called a *potential function* for the vector field F.

Suppose F is continuously differentiable in an open region. The following are equivalent:

def

- F is *conservative* — F is a gradient field
- $F = \nabla f$
- $\int_C F \cdot d\mathbf{r} = \int_a^b f\,(\mathbf{C}(t))||\mathbf{C}'(t)||dt$ is independent of the path taken from $\mathbf{r}(a)$ to $\mathbf{r}(b)$. That is $\int_C F \cdot d\mathbf{r} = \int_D F \cdot d\mathbf{r}$ for any paths C, D that begin at $\mathbf{r}(a)$ and end at $\mathbf{r}(b)$.
- $\int_C F \cdot d\mathbf{r} = f(\mathbf{q}) - f(\mathbf{p})$ where the path C begins at $\mathbf{p}$ and ends at $\mathbf{q}$
- If $F = (P(x,y), Q(x,y))$, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

r(b)

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Method 14.3 Check if F has path independent line integrals

submethod 1: Find f such that $\nabla f = F$

$F(x, y) = (x, y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = \nabla f = \nabla(\frac{1}{2}x^2 + \frac{1}{2}y^2)$.

I.e, $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. ⟵ takes some work

$\int_C F \cdot d\mathbf{r} = f(1, 1) - f(0, 0) = \frac{1}{2} + \frac{1}{2} - 0 = 1$.

Let $F(x,y) = (x,y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Method 14.3 Check if F has path independent line integrals

submethod 2: Check if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, where $F = (P, Q)$.

$$\frac{\partial y}{\partial x} = 0 = \frac{\partial x}{\partial y}$$

$$\frac{\partial y}{\partial x} = \frac{\partial x}{\partial y}$$

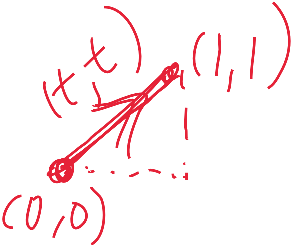
Thus can choose any path starting at (0, 0) and ending at (1, 1)

Ex: Let $\mathbf{C} : [0,1] \to \mathbf{R}^2$, $\mathbf{C}(t) = (t,t)$.

$\int_C F \cdot d\mathbf{r} = \int_0^1 F(\mathbf{C}(t)) \cdot \mathbf{C}'(t)dt$

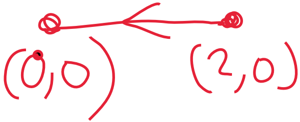
$= \int_0^1 F(t,t) \cdot (1,1)dt$

$= \int_0^1 (t,t) \cdot (1,1)dt = \int_0^1 2tdt = t^2|_0^1 = 1$



Let $F(x,y) = (\overbrace{e^{-y} - 2x}^{P}, \overbrace{-xe^{-y} - sin(y)}^{Q}) = (P(x,y), Q(x,y))$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).



$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} ?$$

Let $F(x, y) = (\overset{P}{e^{-y} - 2x}, \overbrace{-xe^{-y} - sin(y)}^{Q}) = (P(x, y), Q(x, y))$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Does $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$?

$$\frac{\partial(-xe^{-y} - \cancelto{0}{\sin(y)})}{\partial x} = -e^{-y}$$

$$\| \ \checkmark$$

$$\frac{\partial}{\partial y}(e^{-y} - \cancelto{0}{2x}) = -e^{-y}$$

Let $F(x, y) = (e^{-y} - 2x, -xe^{-y} - sin(y)) = (P(x, y), Q(x, y))$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Does $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$?

$$\frac{\partial Q}{\partial x} = \frac{\partial(-xe^{-y} - sin(y))}{\partial x} = -e^{-y} \qquad \frac{\partial P}{\partial y} = \frac{\partial(e^{-y} - 2x)}{\partial y} = -e^{-y}$$

Thus $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

Thus F is a gradient field and hence has path independent integrals.

$F = \nabla f$

Find f

$$F = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$$

Thus $\frac{\partial f}{\partial x} = e^{-y} - 2x$

$$\int \left(\frac{\partial f}{\partial x}(x,y)\right) dx = \int (e^{-y} - 2x)\, dx$$

$$f(x,y) = xe^{-y} - x^2 + c(y)$$

$F = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$

Thus $\frac{\partial f}{\partial x} = e^{-y} - 2x$

Hence $f(x,y) = \int(e^{-y} - 2x)dx = xe^{-y} - x^2 + c(y).$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xe^{-y} - x^2 + c(y)\right)$$

$$= -\cancel{xe^{-y}} + c'(y) = -\cancel{xe^{-y}} - \sin(y)$$

$$c'(y) = -\sin(y)$$

$$F = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$$

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Hence $f(x, y) = \int (e^{-y} - 2x)dx = xe^{-y} - x^2 + c(y)$.

Thus $\frac{\partial f}{\partial y} = \frac{\partial (xe^{-y} - x^2 + c(y))}{\partial y} = -xe^{-y} + c'(y)$

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Thus $\frac{\partial f}{\partial y} = \frac{\partial(xe^{-y} - x^2 + c(y))}{\partial y} = -xe^{-y} + c'(y)$

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Thus $-xe^{-y} - sin(y) = \frac{\partial f}{\partial y} = -xe^{-y} + c'(y)$.

Thus $c'(y) = -sin(y)$ and $c(y) = \int(-sin(y))dy = cos(y) + k$

$\int dy \quad dy$

$f(x,y) = xe^{-y} - x^2 + cos(y) + k$

$$F = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$$

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Thus $c'(y) = -sin(y)$ and $c(y) = \int (-sin(y))dy = cos(y) + k$

Hence $f(x, y) = xe^{-y} - x^2 + cos(y) + k$

Since F is a gradient field, $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$
$= f(0, 0) - f(2, 0) = 0 - 0 + 1 + k - [2 - 4 + 1 + k] = 2$

(0,0) (2,0)

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Alternatively,

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Alternatively, let $X : [0, 2] \to \mathbf{R}^2$, $X(t) = (2 - t, 0) = (x(t), y(t))$. Then $X(0) = (2, 0)$ and $X(2) = (0, 0)$

$x(t) = 2 - t$ implies $dx = -dt$

$y(t) = 0$ implies $dy = 0$.

$\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Alternatively, let $X : [0,2] \to \mathbf{R}^2$, $X(t) = (2-t, 0) = (x(t), y(t))$. Then $X(0) = (2,0)$ and $X(2) = (0,0)$

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$\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$

$= \int_0^2 (1 - 2(2-t))(-dt) = \int_0^2 (3 - 2t)dt = 3t - t^2|_0^2 = 6 - 4 = 2.$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (2, 0).

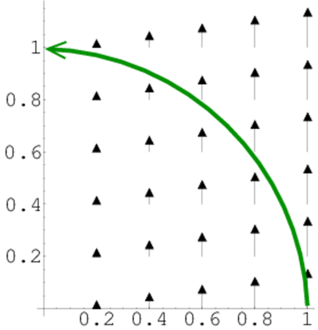
Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (2, 0).

$\int_C F \cdot d\mathbf{r} = 0$ since C is a closed curve and F is conservative.

If a force is given by $\mathbf{F}(x, y) = (0, x)$, compute the work done by the force field on a particle that moves along the curve C that is the counterclockwise quarter unit circle in the first quadrant. In the below picture, the curve C is plotted by th long green curved arrow. The vector field $\mathbf{F}$ is represented by the vertical black arrows.

Choose a parametrization of the curve:

$$\mathbf{c}(t) = (\cos t, \sin t), \quad 0 \le t \le \frac{\pi}{2}.$$

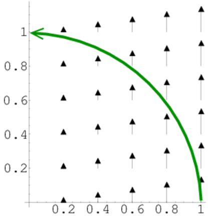


ıttps://mathinsight.org/line_integral_vector_examples

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_0^{\pi/2} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt \\
&= \int_0^{\pi/2} \mathbf{F}(\cos t, \sin t) \cdot (-\sin t, \cos t)dt \\
&= \int_0^{\pi/2} (0, \cos t) \cdot (-\sin t, \cos t)dt \\
&= \int_0^{\pi/2} \cos^2 t\, dt \\
&= \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2t)dt \\
&= \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)\Bigg|_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}.
\end{aligned}
$$

https://mathinsight.org/line_integral_vector_examples

Suppose $F(x,y) = (0,x) = (P,Q)$



$\frac{\partial Q}{\partial x} =$

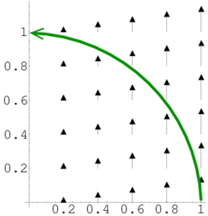
$\frac{\partial P}{\partial y} =$

Suppose $F(x, y) = (0, x) = (P, Q)$

$\frac{\partial Q}{\partial x} =$

$\frac{\partial P}{\partial y} =$

Thus $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$.



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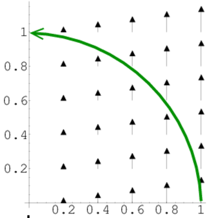

$\frac{\partial Q}{\partial x} =$

$\frac{\partial P}{\partial y} =$

Thus $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$. Thus $\int_D F \cdot d\mathbf{r}$ depends on the path.

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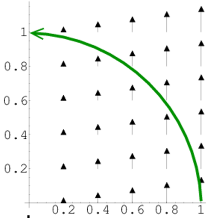

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Let $D(t) =$path from (1, 0) to (0, 1) traveling first along x axis to (0, 0) and then traveling vertically to (1, 1).

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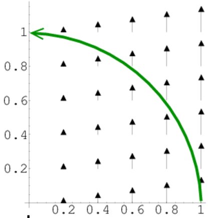

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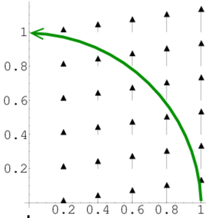
$\int_D F \cdot d\mathbf{r} = 0 + \int 0 \cdot d\mathbf{r} = 0$

Suppose $F(x, y) = (0, x) = (P, Q)$



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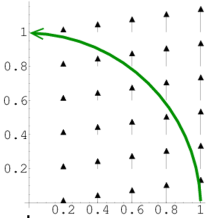
$\int_D F \cdot d\mathbf{r} = 0 + \int 0 \cdot d\mathbf{r} = 0$

Let $E(t) =$path from (1, 0) to (0, 1) traveling first vertically to (1, 1) and then traveling horizontally to (0, 0).

$\int_E F \cdot d\mathbf{r} =$

Suppose $F(x, y) = (0, x) = (P, Q)$



$\frac{\partial Q}{\partial x} =$

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Thus $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$. Thus $\int_D F \cdot d\mathbf{r}$ depends on the path.

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$\int_D F \cdot d\mathbf{r} = 0 + \int 0 \cdot d\mathbf{r} = 0$

Let $E(t)$ =path from (1, 0) to (0, 1) traveling first vertically to (1, 1) and then traveling horizontally to (0, 0).

$\int_E F \cdot d\mathbf{r} = \int_0^1 (0, 1) \cdot (0, 1) dt + 0 = 1$