14.2 Scalar Line Integrals:

Let $\mathbf{C} : [a, b] \to \mathbf{R}^n$ be a smooth curve. $f : \mathbf{R}^n \to R$, a scalar field.

$\Delta s_k =$ length of kth segment of path
$= \int_{t_{k-1}}^{t_k} ||\mathbf{C}'(t)||dt = ||\mathbf{C}'(t_k^*)||(t_k - t_{k-1}) = ||\mathbf{C}'(t_k^*)||\Delta t_k$
for some $t_k^{**} \in [t_{k-1}, t_k]$

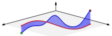
$\int_{\mathbf{C}} f \; d\mathbf{r} \sim \Sigma_{i=1}^n f \; (\mathbf{C}(t_k^*))\Delta s_k = \Sigma_{i=1}^n f(\mathbf{C}(t_k^*))||\mathbf{C}'(t_k^{**})||\Delta t_k$

Thus $\int_{\mathbf{C}} f \; ds = \int_a^b f \; (\mathbf{C}(t))||\mathbf{C}'(t)||dt$

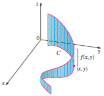
$\int_{\mathbf{C}} f \; ds =$ area under curve $f(\mathbf{C})$

where area is positive above xy plane

and negative below xy plane



https://en.wikipedia.org/wiki/Line_integral



https://brilliant.org/wiki/line-integral

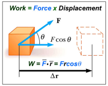
14.2 Vector Line integrals:

Let $\mathbf{C} : [a, b] \to \mathbf{R}^n$ be a smooth path. $F : \mathbf{R}^n \to R^n$, a vector field.

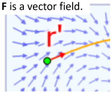
$\mathbf{C}'(t_k^*) \sim \frac{\Delta \mathbf{C}_k}{\Delta t_k}$

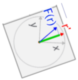
$\int_{\mathbf{C}} F \cdot d\mathbf{r} \sim \Sigma_{i=1}^n F(\mathbf{C}(t_k^*)) \cdot \Delta \mathbf{C}_k = \Sigma_{i=1}^n F(\mathbf{C}(t_k^*)) \cdot \mathbf{C}'(t_k^*)\Delta t_k$

Thus $\int_{\mathbf{C}} F \cdot d\mathbf{r} = \int_a^b F(\mathbf{C}(t)) \cdot \mathbf{C}'(t)dt$



F is a vector field.





http://physicssimplifiedforyou.blogspot.com/

https://en.wikipedia.org/wiki/Line_integral

Thm: Let $\mathbf{C} : [a, b] \to \mathbf{R}^n$ be a piecewise smooth path and let $\mathbf{D} : [c, d] \to \mathbf{R}^n$ be a reparametrization of $\mathbf{C}$. Then

Scalar line integral:

If $f : \mathbf{R}^n \to \mathbf{R}$ is continuous, then $\int_{\mathbf{D}} f \; ds = \int_{\mathbf{C}} f \; ds$

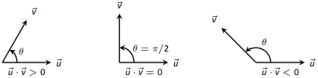
since area under curve does not depend on parametrization.

Vector line integral

If $F : \mathbf{R}^n \to \mathbf{R}^n$ is continuous, then

$\int_{\mathbf{D}} F \cdot d\mathbf{r} = \int_{\mathbf{C}} F \cdot d\mathbf{r}$ if $\mathbf{D}$ is orientation-preserving.

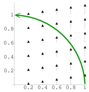
$\int_{\mathbf{D}} F \cdot d\mathbf{r} = -\int_{\mathbf{C}} F \cdot d\mathbf{r}$ if $\mathbf{D}$ is orientation-reversing.

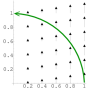


https://math.libretexts.org/Bookshelves/Calculus/Book:_Calculus_(Apex)/10:_Vectors/10.3:_The_Dot_Product

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_{C} \vec{F} \cdot d\vec{r}$$

http://tutorial.math.lamar.edu/Classes/CalcIII/LineIntegralsVectorFields.aspx





Another notation (differential form): For simplicity, we will work in $\mathbf{R}^2$, but the following generalizes to any dimension.

Let $\mathbf{C}(t) = (x(t), y(t))$. Let $F(x,y) = (P(x,y), Q(x,y))$

where $x = x(t), y = y(t)$. Note $x'(t) = \frac{dx}{dt}$, $y'(t) = \frac{dy}{dt}$

Thus $dx = x'(t)dt$ and $dy = y'(t)dt$. Also, $d\mathbf{r} = (dx, dy)$

$$\int_{\mathbf{C}} F \cdot d\mathbf{r} = \int_{\mathbf{C}} (P(x,y), Q(x,y)) \cdot (dx, dy)$$

$$\int_{\mathbf{C}} F \cdot d\mathbf{r} = \int_{\mathbf{C}} P(x,y)dx + Q(x,y)dy$$

$$= \int_a^b P(x(t), y(t))x'(t)dt + Q(x(t), y(t))y'(t)dt$$

Let $F(x,y) = (x,y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Notation 1: Work definition

$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$

$= \int_0^1 F(\mathbf{C}_1(t)) \cdot \mathbf{C}_1'(t)dt + \int_0^1 F(\mathbf{C}_2(t)) \cdot \mathbf{C}_2'(t)dt$

$= \int_0^1 F(t,0) \cdot (1,0)dt + \int_0^1 F(1,t) \cdot (0,1)dt$

$= \int_0^1 (t,0) \cdot (1,0)dt + \int_0^1 (1,t) \cdot (0,1)dt$

$= \int_0^1 tdt + \int_0^1 tdt = 2(\frac{1}{2}t^2)|_0^1 = 1$

Let $F(x,y) = (x,y)$, let $C(t) = C_1(t) \cup C_2(t)$

$C_1(t) = (t,0),\ 0 \le t \le 1$ $\qquad C_2(t) = (1,t),\ 0 \le t \le 1$

$C_1'(t) = (1,0),\ 0 \le t \le 1$ $\qquad C_2'(t) = (0,1),\ 0 \le t \le 1$

Thus along C_1, $dx = 1dt$ and $dy = 0dt$

and along C_2, $dx = 0dt$ and $dy = 1dt$

Notation 2: differential form

$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$
$= \int_{C_1} F \cdot (dx, dy) + \int_{C_2} F \cdot (dx, dy)$

$= \int_{C_1} (x,y) \cdot (dx, dy) + \int_{C_2} (x,y) \cdot (dx, dy)$

$= \int_{C_1} (xdx + ydy) + \int_{C_2} (xdx + ydy)$

$= \int_0^1 (tdt + 0(0dt)) + \int_0^1 (1(0dt) + tdt) = 2(\frac{1}{2}t^2)|_0^1 = 1$

Note: Both of these methods are algebraically equivalent, so it doesn't matter which notation you use.

Let $F(x,y) = (x,y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Can use any parametrization for the path $\mathbf{C}$. For example:

$\mathbf{C}_1 : [0,1] \to \mathbf{R}^2$, $\mathbf{C}_1(t) = (t^2, 0)$

$\mathbf{C}_2 : [0,1] \to \mathbf{R}^2$, $\mathbf{C}_2(t) = (1, t^3)$

Notation 1: Work definition

$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$

$= \int_0^1 F(\mathbf{C}_1(t)) \cdot \mathbf{C}_1'(t)dt + \int_0^1 F(\mathbf{C}_2(t)) \cdot \mathbf{C}_2'(t)dt$

$= \int_0^1 F(t^2,0) \cdot (2t,0)dt + \int_0^1 F(1,t^3) \cdot (0,3t^2)dt$

$= \int_0^1 (t^2,0) \cdot (2t,0)dt + \int_0^1 (1,t^3) \cdot (0,3t^2)dt$

$= \int_0^1 2t^3dt + \int_0^1 3t^5dt = \frac{1}{2}t^4|_0^1 + \frac{1}{2}t^6|_0^1 = 1$

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Can use any parametrization for the path $\mathbf{C}$. For example:

$\mathbf{C}_1 : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}_1(t) = (t^2, 0)$

$\mathbf{C}_2 : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}_2(t) = (1, t^3)$

Notation 2: differential form

$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$

$= \int_{C_1} F \cdot (dx, dy) + \int_{C_2} F \cdot (dx, dy)$

$= \int_{C_1} (x, y) \cdot (dx, dy) + \int_{C_2} (x, y) \cdot (dx, dy)$

$= \int_{C_1} [xdx + ydy] + \int_{C_2} [xdx + ydy]$

$= \int_0^1 t^2(2t)dt + \int_0^1 1(0) + t^3 3t^2 dt = \frac{1}{2}t^4|_0^1 + \frac{1}{2}t^6|_0^1 = 1$

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Methods for special cases:

Method: 14.3 Suppose $F = \nabla f$

Claim F has path independent line integrals.

Method: 14.4 For closed curves, can use Green's Theorem

A path $\mathbf{C} : [a, b] \to \mathbf{R}^n$ is *closed* if $\mathbf{C}(a) = \mathbf{C}(b)$.

If curve is not closed, canNOT use Green's Theorem.

In Calculus I we had the **Fundamental Theorem of Calculus** that told us how to evaluate definite integrals. This told us,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Suppose that C is a **smooth** curve given by $\vec{r}(t)$, $a \le t \le b$. Also suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then,

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

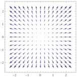
http://tutorial.math.lamar.edu/Classes/CalcIII/FundThmLineIntegrals.aspx

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}[f(\vec{r}(t))]\,dt \quad \text{by the chain rule}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \text{by FTC}$$

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

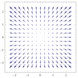


Find f such that $\nabla f = F(x, y) = (x, y)$

If $F(x, y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

$\frac{\partial f(x,y)}{\partial x} = x$ implies $\frac{\partial f(x,y)}{\partial x} \quad = \quad x$

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

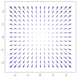


If $F(x, y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $f(x, y) = \frac{x^2}{2} + k(y)$

$\frac{\partial f}{\partial y} =$

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).



If $F(x, y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

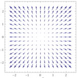
$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $f(x, y) = \frac{x^2}{2} + k(y)$

$\frac{\partial f}{\partial y} = \frac{\partial(\frac{x^2}{2} + k(y))}{\partial y} = k'(y) = y$. Hence $k(y) = \frac{y^2}{2}+$ constant

Thus if we let $f(x, y) =$

then $\nabla f =$

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).



If $F(x, y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $f(x, y) = \frac{x^2}{2} + k(y)$

$\frac{\partial f}{\partial y} = \frac{\partial(\frac{x^2}{2} + k(y))}{\partial y} = k'(y) = y$. Hence $k(y) = \frac{y^2}{2}+$ constant

Thus if we let $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$, then $\nabla f = (x, y) = F$.

Hence $\int_C F \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$$= f(1, 1) - f(0, 0) = \frac{1^2}{2} + \frac{1^2}{2} - 0 = 1$$

F is called *conservative* if $F = \nabla f$. In this case f is called a *potential function* for the vector field F.

Suppose F is continuously differentiable in an open region. The following are equivalent:

- F is *conservative*
- $F = \nabla f$
- $\int_C F \cdot d\mathbf{r} = \int_a^b f\ (\mathbf{C}(t))||\mathbf{C}'(t)||dt$ is independent of the path taken from $\mathbf{r}(a)$ to $\mathbf{r}(b)$. That is $\int_C F \cdot d\mathbf{r} = \int_D F \cdot d\mathbf{r}$ for any paths C, D that begin at $\mathbf{r}(a)$ and end at $\mathbf{r}(b)$.
- $\int_C F \cdot d\mathbf{r} = f(\mathbf{q}) - f(\mathbf{p})$ where the path C begins at $\mathbf{p}$ and ends at $\mathbf{q}$
- If $F = (P(x, y), Q(x, y))$, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Method 14.3 Check if F has path independent line integrals

submethod 1: Find f such that $\nabla f = F$

$F(x, y) = (x, y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = \nabla f = \nabla(\frac{1}{2}x^2 + \frac{1}{2}y^2)$.

I.e, $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$.

$\int_C F \cdot d\mathbf{r} = f(1, 1) - f(0, 0) = \frac{1}{2} + \frac{1}{2} - 0 = 1$.

Let $F(x, y) = (x, y)$, let $C(t)$ =path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Method 14.3 Check if F has path independent line integrals

submethod 2: Check if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, where $F = (P, Q)$.

$\frac{\partial y}{\partial x} = 0 = \frac{\partial x}{\partial y}$

Thus can choose any path starting at (0, 0) and ending at (1, 1)

Ex: Let $\mathbf{C} : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}(t) = (t, t)$.

$\int_C F \cdot d\mathbf{r} = \int_0^1 F(\mathbf{C}(t)) \cdot \mathbf{C}'(t)dt$

$= \int_0^1 F(t, t) \cdot (1, 1)dt$

$= \int_0^1 (t, t) \cdot (1, 1)dt = \int_0^1 2tdt = t^2|_0^1 = 1$

Let $F(x, y) = (e^{-y} - 2x, -xe^{-y} - sin(y)) = (P(x, y), Q(x, y))$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Does $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$?

$\frac{\partial Q}{\partial x} = \frac{\partial(-xe^{-y} - sin(y))}{\partial x} = -e^{-y}$ $\qquad$ $\frac{\partial P}{\partial y} = \frac{\partial(e^{-y} - 2x)}{\partial y} = -e^{-y}$

Thus $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

Thus F is a gradient field and hence has path independent integrals.

$F = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$

Thus $\frac{\partial f}{\partial x} = e^{-y} - 2x$

Hence $f(x, y) = \int (e^{-y} - 2x)dx = xe^{-y} - x^2 + c(y)$.

Thus $\frac{\partial f}{\partial y} = \frac{\partial (xe^{-y} - x^2 + c(y))}{\partial y} = -xe^{-y} + c'(y)$

Thus $-xe^{-y} - sin(y) = \frac{\partial f}{\partial y} = -xe^{-y} + c'(y)$.

Thus $c'(y) = -sin(y)$ and $c(y) = \int (-sin(y))dy = cos(y) + k$

Hence $f(x, y) = xe^{-y} - x^2 + cos(y) + k$

Since F is a gradient field, $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$
$= f(0, 0) - f(2, 0) = 0 - 0 + 1 + k - [2 - 4 + 1 + k] = 2$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Alternatively, let $X : [0, 2] \to \mathbf{R}^2$, $X(t) = (2 - t, 0) = (x(t), y(t))$.
Then $X(0) = (2, 0)$ and $X(2) = (0, 0)$

$x(t) = 2 - t$ implies $dx = -dt$

$y(t) = 0$ implies $dy = 0$.

$\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$

$= \int_0^2 (1 - 2(2 - t))(-dt) = \int_0^2 (3 - 2t)dt = 3t - t^2|_0^2 = 6 - 4 = 2.$

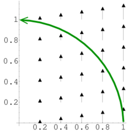
Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (2, 0).

$\int_C F \cdot d\mathbf{r} = 0$ since C is a closed curve and F is conservative.

If a force is given by $\mathbf{F}(x, y) = (0, x)$,
compute the work done by the force field on a particle that moves along the curve C that is the counterclockwise quarter unit circle in the first quadrant. In the below picture, the curve C is plotted by th long green curved arrow. The vector field $\mathbf{F}$ is represented by the vertical black arrows.

Choose a parametrization of the curve:

$\mathbf{c}(t) = (\cos t, \sin t), \quad 0 \leq t \leq \frac{\pi}{2}.$

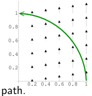


https://mathinsight.org/line_integral_vector_examples

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_0^{\pi/2} \mathbf{F}(\mathbf{c}(t))\cdot \mathbf{c}'(t)dt$$
$$= \int_0^{\pi/2} \mathbf{F}(\cos t, \sin t)\cdot(-\sin t, \cos t)dt$$
$$= \int_0^{\pi/2} (0, \cos t)\cdot(-\sin t, \cos t)dt$$
$$= \int_0^{\pi/2} \cos^2 t\, dt$$
$$= \int_0^{\pi/2} \frac{1}{2}(1+\cos 2t)dt$$
$$= \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)\Bigg|_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}.$$

https://mathinsight.org/line_integral_vector_examples

Suppose $F(x, y) = (0, x) = (P, Q)$



$\frac{\partial Q}{\partial x} =$

$\frac{\partial P}{\partial y} =$

Thus $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$. Thus $\int_D F\cdot d\mathbf{r}$ depends on the path.

Let $D(t)$ =path from (1, 0) to (0, 1) traveling first along x axis to (0, 0) and then traveling vertically to (1, 1).

$\int_D F\cdot d\mathbf{r} = 0 + \int 0\cdot d\mathbf{r} = 0$

Let $E(t)$ =path from (1, 0) to (0, 1) traveling first vertically to (1, 1) and then traveling horizontally to (0, 0).

$\int_E F\cdot d\mathbf{r} = \int_0^1 (0,1)\cdot(0,1)dt + 0 = 1$