14.2 Scalar Line Integrals:

Let $\mathbf{C} : [a, b] \to \mathbf{R}^n$ be a smooth curve. $f : \mathbf{R}^n \to R$, a scalar field.

$\Delta s_k =$ length of kth segment of path

$$= \int_{t_{k-1}}^{t_k} ||\mathbf{C}'(t)||dt = ||\mathbf{C}'(t_k^*)||(t_k - t_{k-1}) = ||\mathbf{C}'(t_k^*)||\Delta t_k$$

for some $t_k^{**} \in [t_{k-1}, t_k]$

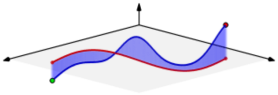
$$\int_{\mathbf{C}} f \; d\mathbf{r} \sim \Sigma_{i=1}^n f \; (\mathbf{C}(t_k^*))\Delta s_k = \Sigma_{i=1}^n f(\mathbf{C}(t_k^*))||\mathbf{C}'(t_k^{**})||\Delta t_k$$

Thus $\int_{\mathbf{C}} f \; ds = \int_a^b f \; (\mathbf{C}(t))||\mathbf{C}'(t)||dt$

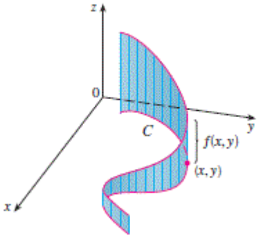
$\int_{\mathbf{C}} f \; ds =$ area under curve $f(\mathbf{C})$

where area is positive above xy plane

and negative below xy plane



https://en.wikipedia.org/wiki/Line_integral



https://brilliant.org/wiki/line-integral

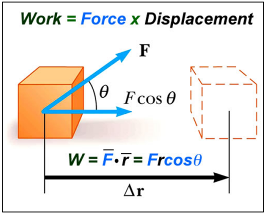
14.2 Vector Line integrals:

Let $\mathbf{C} : [a, b] \to \mathbf{R}^n$ be a smooth path. $F : \mathbf{R}^n \to R^n$, a vector field.

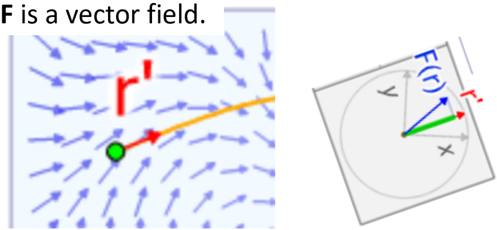
$$\mathbf{C}'(t_k^*) \sim \frac{\Delta \mathbf{C}_k}{\Delta t_k}$$

$$\int_{\mathbf{C}} F \cdot d\mathbf{r} \sim \Sigma_{i=1}^{n} F(\mathbf{C}(t_k^*)) \cdot \Delta \mathbf{C}_k = \Sigma_{i=1}^{n} F(\mathbf{C}(t_k^*)) \cdot \mathbf{C}'(t_k^*) \Delta t_k$$

$$\text{Thus } \int_{\mathbf{C}} F \cdot d\mathbf{r} = \int_a^b F(\mathbf{C}(t)) \cdot \mathbf{C}'(t) dt$$



http://physicssimplifiedforyou.blogspot.com/



https://en.wikipedia.org/wiki/Line_integral

Thm: Let $\mathbf{C} : [a, b] \to \mathbf{R}^n$ be a piecewise smooth path and let $\mathbf{D} : [c, d] \to \mathbf{R}^n$ be a reparametrization of $\mathbf{C}$. Then

Scalar line integral:

If $f : \mathbf{R}^n \to \mathbf{R}$ is continuous, then $\int_{\mathbf{D}} f \ ds = \int_{\mathbf{C}} f \ ds$

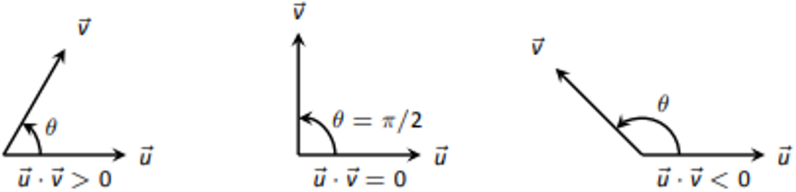
since area under curve does not depend on parametrization.

Vector line integral

If $F : \mathbf{R}^n \to \mathbf{R}^n$ is continuous, then

$\int_{\mathbf{D}} F \cdot d\mathbf{r} = \int_{\mathbf{C}} F \cdot d\mathbf{r}$ if $\mathbf{D}$ is orientation-preserving.

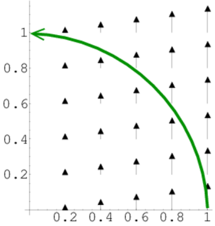
$\int_{\mathbf{D}} F \cdot d\mathbf{r} = -\int_{\mathbf{C}} F \cdot d\mathbf{r}$ if $\mathbf{D}$ is orientation-reversing.

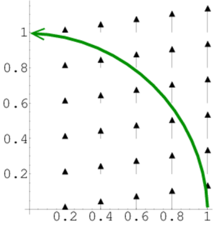


https://math.libretexts.org/Bookshelves/Calculus/Book:_Calculus_(Apex)/10:_Vectors/10.3:_The_Dot_Product

$$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_{C} \vec{F}\cdot d\vec{r}$$

http://tutorial.math.lamar.edu/Classes/CalcIII/LineIntegralsVectorFields.aspx





Another notation (differential form): For simplicity, we will work in $\mathbf{R}^2$, but the following generalizes to any dimension.

Let $\mathbf{C}(t) = (x(t), y(t))$. Let $F(x, y) = (P(x, y), Q(x, y))$

where $x = x(t), y = y(t)$. Note $x'(t) = \frac{dx}{dt}$, $y'(t) = \frac{dy}{dt}$

Thus $dx = x'(t)dt$ and $dy = y'(t)dt$. Also, $d\mathbf{r} = (dx, dy)$

$$\int_{\mathbf{C}} F \cdot d\mathbf{r} = \int_{\mathbf{C}} (P(x, y), Q(x, y)) \cdot (dx, dy)$$

$$\int_{\mathbf{C}} F \cdot d\mathbf{r} = \int_{\mathbf{C}} P(x, y)dx + Q(x, y)dy$$

$$= \int_a^b P(x(t), y(t))x'(t)dt + Q(x(t), y(t))y'(t)dt$$

Let $F(x, y) = (x, y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Notation 1: Work definition

$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$

$= \int_0^1 F(\mathbf{C}_1(t)) \cdot \mathbf{C}_1'(t)dt + \int_0^1 F(\mathbf{C}_2(t)) \cdot \mathbf{C}_2'(t)dt$

$= \int_0^1 F(t, 0) \cdot (1, 0)dt + \int_0^1 F(1, t) \cdot (0, 1)dt$

$= \int_0^1 (t, 0) \cdot (1, 0)dt + \int_0^1 (1, t) \cdot (0, 1)dt$

$= \int_0^1 tdt + \int_0^1 tdt = 2(\frac{1}{2}t^2)|_0^1 = 1$

Let $F(x, y) = (x, y)$, let $C(t) = C_1(t) \cup C_2(t)$

$C_1(t) = (t, 0),\ \ 0 \leq t \leq 1$ $\qquad$ $C_2(t) = (1, t),\ \ 0 \leq t \leq 1$

$C_1'(t) = (1, 0),\ \ 0 \leq t \leq 1$ $\qquad$ $C_2'(t) = (0, 1),\ \ 0 \leq t \leq 1$

Thus along C_1, $dx = 1dt$ and $dy = 0dt$

and along C_2, $dx = 0dt$ and $dy = 1dt$

Notation 2: differential form

$$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$$
$$= \int_{C_1} F \cdot (dx, dy) + \int_{C_2} F \cdot (dx, dy)$$

$$= \int_{C_1} (x, y) \cdot (dx, dy) + \int_{C_2} (x, y) \cdot (dx, dy)$$

$$= \int_{C_1} (xdx + ydy) + \int_{C_2} (xdx + ydy)$$

$$= \int_0^1 (tdt + 0(0dt)) + \int_0^1 (1(0dt) + tdt) = 2(\tfrac{1}{2}t^2)|_0^1 = 1$$

Note: Both of these methods are algebraically equivalent, so it doesn't matter which notation you use.

Let $F(x, y) = (x, y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Can use any parametrization for the path $\mathbf{C}$. For example:

$\mathbf{C}_1 : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}_1(t) = (t^2, 0)$

$\mathbf{C}_2 : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}_2(t) = (1, t^3)$

Notation 1: Work definition

$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$

$= \int_0^1 F(\mathbf{C}_1(t)) \cdot \mathbf{C}_1'(t)dt + \int_0^1 F(\mathbf{C}_2(t)) \cdot \mathbf{C}_2'(t)dt$

$= \int_0^1 F(t^2, 0) \cdot (2t, 0)dt + \int_0^1 F(1, t^3) \cdot (0, 3t^2)dt$

$= \int_0^1 (t^2, 0) \cdot (2t, 0)dt + \int_0^1 (1, t^3) \cdot (0, 3t^2)dt$

$= \int_0^1 2t^3 dt + \int_0^1 3t^5 dt = \frac{1}{2}t^4|_0^1 + \frac{1}{2}t^6|_0^1 = 1$

Let $F(x, y) = (x, y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Can use any parametrization for the path $\mathbf{C}$. For example:

$\mathbf{C}_1 : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}_1(t) = (t^2, 0)$

$\mathbf{C}_2 : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}_2(t) = (1, t^3)$

Notation 2: differential form

$\int_C F \cdot d\mathbf{r} = \int_{C_1} F \cdot d\mathbf{r} + \int_{C_2} F \cdot d\mathbf{r}$

$= \int_{C_1} F \cdot (dx, dy) + \int_{C_2} F \cdot (dx, dy)$

$= \int_{C_1} (x, y) \cdot (dx, dy) + \int_{C_2} (x, y) \cdot (dx, dy)$

$= \int_{C_1} [xdx + ydy] + \int_{C_2} [xdx + ydy]$

$= \int_0^1 t^2(2t)dt + \int_0^1 1(0) + t^3 3t^2 dt = \frac{1}{2}t^4|_0^1 + \frac{1}{2}t^6|_0^1 = 1$

Let $F(x, y) = (x, y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Methods for special cases:

Method: 14.3 Suppose $F = \nabla f$

Claim F has path independent line integrals.

Method: 14.4 For closed curves, can use Green's Theorem

A path $\mathbf{C} : [a, b] \to \mathbf{R}^n$ is *closed* if $\mathbf{C}(a) = \mathbf{C}(b)$.

If curve is not closed, canNOT use Green's Theorem.

In Calculus I we had the **Fundamental Theorem of Calculus** that told us how to evaluate definite integrals. This told us,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Suppose that C is a **smooth** curve given by $\vec{r}(t)$, $a \le t \le b$. Also suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then,

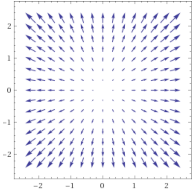
$$\int\limits_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\int\limits_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f\left(\vec{r}(t)\right) \cdot \vec{r}'(t)\; dt$$

$$= \int_a^b \frac{d}{dt}\left[f\left(\vec{r}(t)\right)\right] dt \quad \text{by the chain rule}$$

$$= f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right) \quad \text{by FTC}$$

Let $F(x,y) = (x,y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

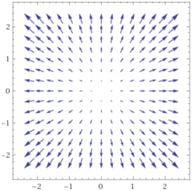


Find f such that $\nabla f = F(x,y) = (x,y)$

If $F(x,y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

$\frac{\partial f(x,y)}{\partial x} = x$ implies $\quad \frac{\partial f(x,y)}{\partial x} \quad = \quad x$

Let $F(x,y) = (x,y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

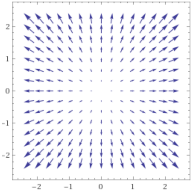


If $F(x,y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $f(x,y) = \frac{x^2}{2} + k(y)$

$\frac{\partial f}{\partial y} =$

Let $F(x,y) = (x,y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).



If $F(x,y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

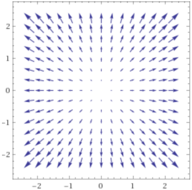
$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $f(x,y) = \frac{x^2}{2} + k(y)$

$\frac{\partial f}{\partial y} = \frac{\partial(\frac{x^2}{2} + k(y))}{\partial y} = k'(y) = y$. Hence $k(y) = \frac{y^2}{2} +$ constant

Thus if we let $f(x,y) =$

then $\nabla f =$

Let $F(x,y) = (x,y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).



If $F(x,y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

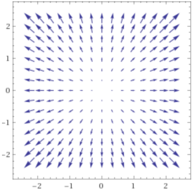
$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $f(x,y) = \frac{x^2}{2} + k(y)$

$\frac{\partial f}{\partial y} = \frac{\partial(\frac{x^2}{2} + k(y))}{\partial y} = k'(y) = y$. Hence $k(y) = \frac{y^2}{2} +$ constant

Thus if we let $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$, then $\nabla f = (x,y) = F$.

Hence $\int_C F \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

Let $F(x,y) = (x,y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).



If $F(x,y) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then

$\frac{\partial f(x,y)}{\partial x} = x$ implies $\int \frac{\partial f(x,y)}{\partial x} dx = \int x dx$ implies $f(x,y) = \frac{x^2}{2} + k(y)$

$\frac{\partial f}{\partial y} = \frac{\partial(\frac{x^2}{2} + k(y))}{\partial y} = k'(y) = y$. Hence $k(y) = \frac{y^2}{2}+$ constant

Thus if we let $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$, then $\nabla f = (x,y) = F$.

Hence $\int_C F \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$$= f(1,1) - f(0,0) = \frac{1^2}{2} + \frac{1^2}{2} - 0 = 1$$

F is called *conservative* if $F = \nabla f$. In this case f is called a *potential function* for the vector field F.

Suppose F is continuously differentiable in an open region. The following are equivalent:

- F is *conservative*
- $F = \nabla f$
- $\int_C F \cdot d\mathbf{r} = \int_a^b f\ (\mathbf{C}(t))||\mathbf{C}'(t)||dt$ is independent of the path taken from $\mathbf{r}(a)$ to $\mathbf{r}(b)$. That is $\int_C F \cdot d\mathbf{r} = \int_D F \cdot d\mathbf{r}$ for any paths C, D that begin at $\mathbf{r}(a)$ and end at $\mathbf{r}(b)$.
- $\int_C F \cdot d\mathbf{r} = f(\mathbf{q}) - f(\mathbf{p})$ where the path C begins at $\mathbf{p}$ and ends at $\mathbf{q}$
- If $F = (P(x,y), Q(x,y))$, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Let $F(x, y) = (x, y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Method 14.3 Check if F has path independent line integrals

submethod 1: Find f such that $\nabla f = F$

$F(x, y) = (x, y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = \nabla f = \nabla(\frac{1}{2}x^2 + \frac{1}{2}y^2)$.

I.e, $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$.

$\int_C F \cdot d\mathbf{r} = f(1, 1) - f(0, 0) = \frac{1}{2} + \frac{1}{2} - 0 = 1$.

Let $F(x, y) = (x, y)$, let $C(t) =$path from (0, 0) to (1, 1) traveling first along x axis to (1, 0) and then traveling vertically to (1, 1).

Method 14.3 Check if F has path independent line integrals

submethod 2: Check if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, where $F = (P, Q)$.

$$\frac{\partial y}{\partial x} = 0 = \frac{\partial x}{\partial y}$$

Thus can choose any path starting at (0, 0) and ending at (1, 1)

Ex: Let $\mathbf{C} : [0, 1] \to \mathbf{R}^2$, $\mathbf{C}(t) = (t, t)$.

$$\int_C F \cdot d\mathbf{r} = \int_0^1 F(\mathbf{C}(t)) \cdot \mathbf{C}'(t) dt$$

$$= \int_0^1 F(t, t) \cdot (1, 1) dt$$

$$= \int_0^1 (t, t) \cdot (1, 1) dt = \int_0^1 2t dt = t^2|_0^1 = 1$$

Let $F(x, y) = (e^{-y} - 2x, -xe^{-y} - sin(y)) = (P(x, y), Q(x, y))$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Let $F(x, y) = (e^{-y} - 2x, -xe^{-y} - sin(y)) = (P(x, y), Q(x, y))$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Does $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$?

Let $F(x, y) = (e^{-y} - 2x, -xe^{-y} - sin(y)) = (P(x, y), Q(x, y))$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Does $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$?

$\frac{\partial Q}{\partial x} = \frac{\partial(-xe^{-y} - sin(y))}{\partial x} = -e^{-y}$

$\frac{\partial P}{\partial y} = \frac{\partial(e^{-y} - 2x)}{\partial y} = -e^{-y}$

Thus $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

Thus F is a gradient field and hence has path independent integrals.

$$F = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$$

Thus $\frac{\partial f}{\partial x} = e^{-y} - 2x$

$$F = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$$

Thus $\frac{\partial f}{\partial x} = e^{-y} - 2x$

Hence $f(x, y) = \int (e^{-y} - 2x)dx = xe^{-y} - x^2 + c(y)$.

$$F = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$$

Thus $\frac{\partial f}{\partial x} = e^{-y} - 2x$

Hence $f(x, y) = \int (e^{-y} - 2x)dx = xe^{-y} - x^2 + c(y)$.

Thus $\frac{\partial f}{\partial y} = \frac{\partial (xe^{-y} - x^2 + c(y))}{\partial y} = -xe^{-y} + c'(y)$

$F = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$

Thus $\frac{\partial f}{\partial x} = e^{-y} - 2x$

Hence $f(x, y) = \int (e^{-y} - 2x)dx = xe^{-y} - x^2 + c(y)$.

Thus $\frac{\partial f}{\partial y} = \frac{\partial (xe^{-y} - x^2 + c(y))}{\partial y} = -xe^{-y} + c'(y)$

Thus $-xe^{-y} - sin(y) = \frac{\partial f}{\partial y} = -xe^{-y} + c'(y)$.

$F = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$

Thus $\frac{\partial f}{\partial x} = e^{-y} - 2x$

Hence $f(x, y) = \int (e^{-y} - 2x)dx = xe^{-y} - x^2 + c(y)$.

Thus $\frac{\partial f}{\partial y} = \frac{\partial (xe^{-y} - x^2 + c(y))}{\partial y} = -xe^{-y} + c'(y)$

Thus $-xe^{-y} - sin(y) = \frac{\partial f}{\partial y} = -xe^{-y} + c'(y)$.

Thus $c'(y) = -sin(y)$ and $c(y) = \int (-sin(y))dy = cos(y) + k$

$F = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (P, Q) = (e^{-y} - 2x, -xe^{-y} - sin(y))$

Thus $\frac{\partial f}{\partial x} = e^{-y} - 2x$

Hence $f(x,y) = \int (e^{-y} - 2x)dx = xe^{-y} - x^2 + c(y)$.

Thus $\frac{\partial f}{\partial y} = \frac{\partial (xe^{-y} - x^2 + c(y))}{\partial y} = -xe^{-y} + c'(y)$

Thus $-xe^{-y} - sin(y) = \frac{\partial f}{\partial y} = -xe^{-y} + c'(y)$.

Thus $c'(y) = -sin(y)$ and $c(y) = \int (-sin(y))dy = cos(y) + k$

Hence $f(x,y) = xe^{-y} - x^2 + cos(y) + k$

Since F is a gradient field, $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$
$= f(0,0) - f(2,0) = 0 - 0 + 1 + k - [2 - 4 + 1 + k] = 2$

Find $\int_C (e^{-y} - 2x)dx + (-xe^{-y} - sin(y))dy$, where C is a piecewise smooth curve starting at (2, 0) and ending at (0, 0).

Alternatively,

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$x(t) = 2 - t$ implies $dx = -dt$

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$= \int_0^2 (1 - 2(2 - t))(-dt) = \int_0^2 (3 - 2t)dt = 3t - t^2|_0^2 = 6 - 4 = 2.$

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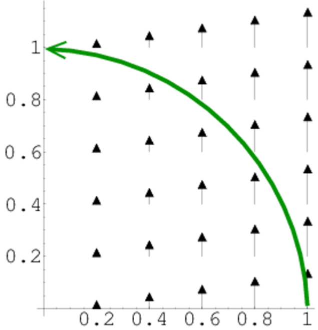
$\int_C F \cdot d\mathbf{r} = 0$ since C is a closed curve and F is conservative.

If a force is given by $\mathbf{F}(x, y) = (0, x)$, compute the work done by the force field on a particle that moves along the curve C that is the counterclockwise quarter unit circle in the first quadrant. In the below picture, the curve C is plotted by th long green curved arrow. The vector field $\mathbf{F}$ is represented by the vertical black arrows.

Choose a parametrization of the curve:

$$\mathbf{c}(t) = (\cos t, \sin t), \quad 0 \leq t \leq \frac{\pi}{2}.$$



ıttps://mathinsight.org/line_integral_vector_examples

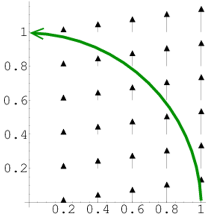
$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_0^{\pi/2} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt \\
&= \int_0^{\pi/2} \mathbf{F}(\cos t, \sin t) \cdot (-\sin t, \cos t)dt \\
&= \int_0^{\pi/2} (0, \cos t) \cdot (-\sin t, \cos t)dt \\
&= \int_0^{\pi/2} \cos^2 t\, dt \\
&= \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2t)dt \\
&= \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)\Bigg|_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}.
\end{aligned}
$$

https://mathinsight.org/line_integral_vector_examples

Suppose $F(x, y) = (0, x) = (P, Q)$

$\frac{\partial Q}{\partial x} =$

$\frac{\partial P}{\partial y} =$



Suppose $F(x, y) = (0, x) = (P, Q)$

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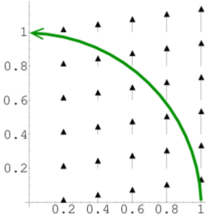
Thus $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$.



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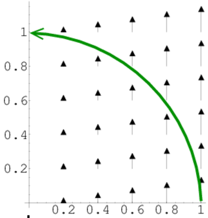

$\frac{\partial Q}{\partial x} =$

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Thus $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$. Thus $\int_D F \cdot d\mathbf{r}$ depends on the path.

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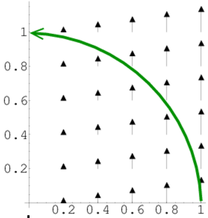

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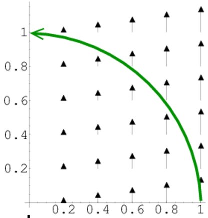

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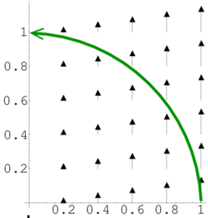

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$\int_E F \cdot d\mathbf{r} =$

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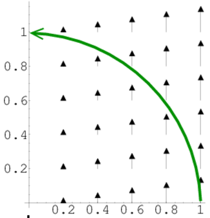

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$\int_E F \cdot d\mathbf{r} = \int_0^1 (0,1) \cdot (0,1) dt + 0 = 1$