$u(x,t) =$ Temperature at point x and time t

$\alpha^2 =$ thermal diffusivity constant

Initial values: $u(x,0) = f(x) \quad$ for $0 \leq x \leq L$.

PDE: $\alpha^2 u_{xx} = u_t \quad$ for $0 \leq x \leq L$ and $t > 0$

Boundary values: $u(0,t) = 0, \quad u(L,t) = 0 \quad$ for $t > 0$

Step 1: Suppose $u(x,t) = X(x)T(t)$ (Note big assumption)

Step 1a: Plug into PDE: $\alpha^2 X''(x)T(t) = X(x)T'(t)$

Separate Variables: $\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}$

Thus we obtain two linear homogeneous ODEs:

$X''(x) + \lambda X(x) = 0$ and $T'(t) + \alpha^2 \lambda T(t) = 0$

Step 1: Suppose $u(x,t) = X(x)T(t)$ (Note big assumption)

Step 1b: Plug into boundary values: $u(0,t) = 0, \quad u(L,t) = 0 \quad$ for $t > 0$

$u(0,t) = X(0)T(t) = 0$ and $u(L,t) = X(L)T(t) = 0, \quad$ for $t > 0$

implies $X(0) = 0$ and $X(L) = 0$

since $T(t) \neq 0 \quad$ for $t > 0$ (looking for non-trivial solns)

. If $T(t) = 0$ for $t > 0$, then $u(x,t) = X(x)T(t) = 0$ for all t, x

A boring solution which might not satisfy initial condition:

$$u(x,0) = f(x) \quad \text{for } 0 \leq x \leq L.$$

Thus $T(t) \neq 0$ for all t and hence $X(0) = 0$

Boundary values: $u(0,t) = 0, \quad u(L,t) = 0 \quad$ for $t > 0$

$u(L,t) = X(L)T(t) = 0, \quad$ for $t > 0$

If $T(t) = 0$ for $t > 0$, then $u(x,t) = X(x)T(t) = 0$ for all t, x

A boring solution which might not satisfy initial condition:

$$u(x,0) = f(x) \quad \text{for } 0 \leq x \leq L.$$

Thus $T(t) \neq 0$ for all t and hence $X(L) = 0$

$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \ X(L) = 0$

$T'(t) + \alpha^2 \lambda T(t) = 0$

The trivial solution $X(x) = 0$ for all x satisfies all homogeneous linear ODE's and also satisfies our boundary conditions. But then

$$u(x,t) = X(x)T(t) = 0 \text{ for all } t, x$$

A boring solution which might not satisfy initial condition:

$$u(x,0) = f(x) \quad \text{for } 0 \leq x \leq L.$$

Solve $T'(t) + \alpha^2 \lambda T(t) = 0$

characteristic equation: $r + \alpha^2 \lambda = 0$

Thus $r = -\alpha^2 \lambda$

Thus $T(t) = Ce^{-\alpha^2 \lambda t}$

By the 2nd order linear homogeneous ODE

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \ldots$$

Thus $T(t) = Ce^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} = Ce^{-t\left(\frac{\alpha n\pi}{L}\right)^2}$

$T(t) = Ce^{-t\left(\frac{\alpha n\pi}{L}\right)^2}$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-t\left(\frac{\alpha n\pi}{L}\right)^2} sin\left(\frac{n\pi x}{L}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n sin\left(\frac{n\pi x}{L}\right) = f(x)$$

Note: $u(x,0)$ is the Fourier sine series for f defined on $[0, L]$