Math 3550 Midterm 1 Name:

March 3, 2020 Show all work. Circle section number: 91 (9:30 am) 131 (1:30pm)

1.) For the plane curve, $x(t) = t$, $y(t) = ln(t)$,

[6] 1a.) State the integral that gives the arclength from $t = 1$ to $t = e$ (you do not need to calculate the integral).

curve $= r(t) = (t, lnt)$

velocity vector $= r'(t) = (1, \frac{1}{t})$

speed = length of velocity vector $= \sqrt{1 + \frac{1}{t^2}}$

Answer: $\int_1^e \sqrt{1 + \frac{1}{t^2}}\ dt$

[8] 1b.) Find the unit tangent and unit normal to this curve at the point (e, 1).

The point $(e, 1)$ occurs at time $t = e$ since $r(e) = (e, ln(e)) = (e, 1)$

velocity vector $= r'(e) = (1, \frac{1}{e})$

speed = length of velocity vector $= \sqrt{1 + \frac{1}{e^2}}$

To create unit vector, divide by length. Note can take any vector that is a positive multiple of the velocity vector. For example $(e, 1) = e(1, \frac{1}{e})$

$\mathbf{T} = \frac{(e,1)}{\sqrt{1+e^2}}$

Answer: $\mathbf{T} = \dfrac{(e,1)}{\sqrt{1+e^2}}$ and $\mathbf{N} = \dfrac{(1,-e)}{\sqrt{1+e^2}}$

2.) Circle T for true and F for false.

[2] 2a.) If $\mathbf{a}$ and $\mathbf{b}$ represent adjacent sides of a parallelogram $PQRS$, so that $\mathbf{a} = \overrightarrow{RQ}$ and $\mathbf{b} = \overrightarrow{RS}$, then the area of $PQRS$ is $|\mathbf{a} \cdot \mathbf{b}|$.

F

[2] 2b.) If $\mathbf{a}$ and $\mathbf{b}$ represent adjacent sides of a parallelogram $PQRS$, so that $\mathbf{a} = \overrightarrow{RQ}$ and $\mathbf{b} = \overrightarrow{RS}$, then the area of $PQRS$ is $|\mathbf{a} \times \mathbf{b}|$.

T

[2] 2c.) If f is integrable, then the Riemann sum $\sum_{i=1}^{k} f(x_i^*, y_i^*)\Delta A_i$ can be made arbitrarily close to the value of the double integral $\int\int_R f(x,y)dA$ by choosing an inner partition of R with sufficiently small norm.

T

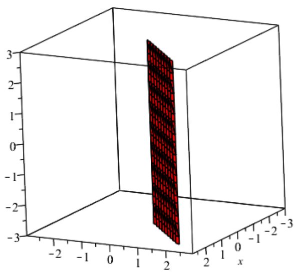
3.) Match the function to its graph by circling the appropriate letter. Recall (r, θ, z) refers to cylindrical coordinates, while (ρ, θ, ϕ) refers to spherical coordinates.

[3] 3i.) $r = 4$ D

[3] 3ii.) $\rho = 4.$ F

[3] 3iii.) $\theta = \frac{\pi}{4}$ A

[3] 3iv.) $\phi = \frac{\pi}{4}$ H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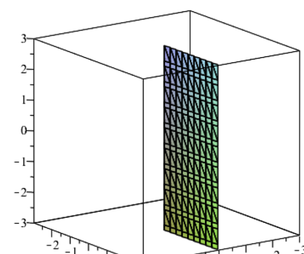
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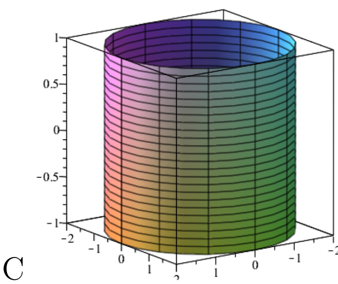
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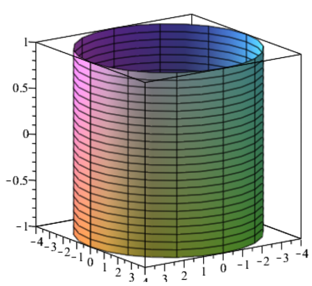
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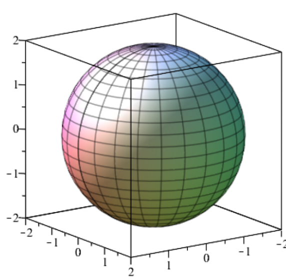
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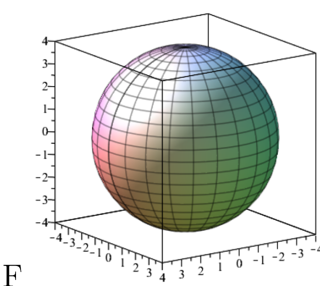
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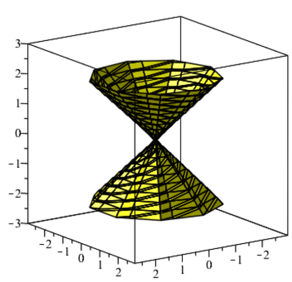
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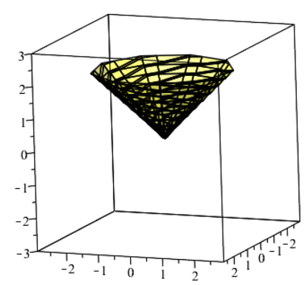
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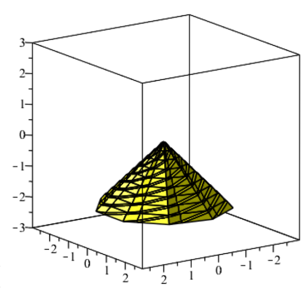
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4.) For the surface $f(x, y) = x^3 - 3x^2 + 3xy^2 - 3y^2$,

[4] 4a.) $\nabla f = \underline{\quad < 3x^2 - 6x + 3y^2, \;\; 6xy - 6y > \quad}$

[2] 4b.) If f represents elevation, the direction of steepest ascent starting at $(x, y) = (1, 2)$ is

$< 3(1)^2 - 6(1) + 3(2)^2, \;\; 6(1)(2) - 6(2) > \;\; = \;\; < 3 - 6 + 12, \;\; 12 - 12 > \;\; = \;\; < 9, \; 0 >$

(or any positive multiple of $< 9, \; 0 >$, for example $< 1, \; 0 >$)

[2] 4c.) If f represents elevation, the direction of steepest descent starting at the point $(x, y) = (1, 2)$ is

$< -9, 0 >$ (or any negative multiple of $< 9, \; 0 >$ such as $< -1, \; 0 >$)

[5] 4d.) The equation of the tangent plane at $(1, 2, -2)$ is $\underline{\quad 9x - z = 11 \quad}$

Normal vector to tangent plane $= \; < \frac{\partial z}{\partial x}(1, 2), \; \frac{\partial z}{\partial y}(1, 2), \; -1 > \;\; = \;\; < 9, \; 0, \; -1 >$

Equation of plane: $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$

$< 9, \; 0, \; -1 > \cdot < x - 1, \; y - 2, \; z + 2 > \;\; = \;\; 9(x - 1) + 0(y - 2) - 1(z + 2) = 0$

$9x - z = 9 + 2 = 11$

[4] 4e.) Use differentials to estimate $f(1.1, 1.8)$.

$9x + 0y - z = 11$ Thus $z = 9x + 0y - 11$

$z(1.1, 1.8) = 9(1.1) + 0(1.8) - 11 = 9 + 0(2) - 11 + 9(0.1) + 0(-0.2)$

$= -2 + 9(0.1) + 0(-0.2) = -2 + 0.9 + 0 = -1.1$

4 continued.) For the surface $f(x, y) = x^3 - 3x^2 + 3xy^2 - 3y^2$,

[8] 4f.) Find and classify the critical points using the 2nd derivative test.

$\nabla f \;=\; < 3x^2 - 6x + 3y^2,\;\; 6xy - 6y > \;=\; < 0, 0 >$

$3x^2 - 6x + 3y^2 = 0$ and $6xy - 6y = 0$

$6xy - 6y^2 = 0$ implies $6y(x - 1) = 0$

Thus $y = 0$ or $x = 1$

If $y = 0$, then $3x^2 - 6x + 3y^2 = 3x^2 - 6x = 3x(x - 2)0$. Thus $x = 0, 2$

If $x = 1$, then $3x^2 - 6x + 3y^2 = 3(1)^2 - 6(1) + 3y^2 = -3 + 3y^2 = 0$. Thus $y^2 = 1$ and $y = \pm 1$

Thus critical points are (0, 0), (2, 0), (1, -1), (1, 1)

$\nabla f \;=\; < 3x^2 - 6x + 3y^2,\;\; 6xy - 6y >$

Thus $det(D^2 f) = \begin{vmatrix} 6x - 6 & 6y \\ 6y & 6x - 6 \end{vmatrix} = (6x - 6)^2 - 36y^2 = 6^2(x - 1)^2 - 36y^2 = 36[(x - 1)^2 - y^2]$

$det(D^2 f(0, 0)) = 36[(0 - 1)^2 - (0)^2] > 0$. Thus local min or max. $f_{xx} = 6(0) - 6 < 0$. Thus local max.

$det(D^2 f(2, 0)) = 36[(2 - 1)^2 - (0)^2] > 0$. Thus local min or max $f_{xx} = 6(2) - 6 > 0$. Thus local min.

$det(D^2 f(1, -1)) = 36[(1 - 1)^2 - (-1)^2] < 0$. Thus saddle.

$det(D^2 f(1, 1)) = 36[(1 - 1)^2 - (1)^2] < 0$ Thus saddle.

The critical point $\underline{\;(0, 0)\;}$ is a $\underline{\;\textit{local maximum}\;}$

The critical point $\underline{\;(2, 0)\;}$ is a $\underline{\;\textit{local minimum}\;}$

The critical point $\underline{\;(1, -1)\;}$ is a $\underline{\;\textit{saddle}\;}$

The critical point $\underline{\;(1, 1)\;}$ is a $\underline{\;\textit{saddle}\;}$

[15] 5.) Use the method of Lagrange multipliers to find the point(s) on the surface $z = 3xy + 6$ closest to the origin.

Since surface is in $\mathbb{R}^3$, we want the point (x, y, z) closest to the origin $(0, 0, 0)$

Minimizing distance squared $= d^2 = f(x, y, z) = (x - 0)^2 + (y - 0)^2 + (z - 0)^2 = x^2 + y^2 + z^2$

subject to constraint $g(x, y, z) = 3xy + 6 - z = 0$

$\nabla f \;=\; < 2x, 2y, 2z >, \qquad \nabla g \;=\; < 3y, 3x, -1 >$

The equation $\nabla g =< 3y, 3x, -1 > \;=\; < 0, 0, 0 >$ has no solution, so just need to solve

$\nabla f \;=\; \lambda \nabla g$

$< 2x, 2y, 2z > \;=\; \lambda < 3y, 3x, -1 >$ subject to constraint $g(x, y, z) = 3xy + 6 - z = 0$

$2x = \lambda(3y), \quad 2y = \lambda(3x), \quad 2z = \lambda(-1)$

Thus $\lambda = -2z$ and $2x = (-2z)(3y), \quad 2y = (-2z)(3x)$

Thus $x = -3zy, \quad y = -3zx = -3z(-3zy) = 9z^2y.$

Thus $y = 9z^2y$ and if $y \neq 0, \;\; z = \pm\frac{1}{3}$

$x = -3zy$ implies $x = -3(\pm\frac{1}{3})y = \mp y$

Or alternatively,

$2x = \lambda(3y), \quad 2y = \lambda(3x), \quad 2z = \lambda(-1)$

$2x^2 = \lambda(3xy), \quad 2y^2 = \lambda(3xy), \quad -6xyz = \lambda(3xy)$

Thus $2x^2 = 2y^2 = -6xyz.$

$2x^2 = 2y^2$ implies $x = \pm y$

$2y^2 = -6xyz = -6(\pm y)yz = \mp 6y^2z$ implies $1 = \mp 3z$. Thus $z = \mp\frac{1}{3}$

Plugging into constraint: $3(\pm y)y + 6 - (\mp\frac{1}{3}) = 0$

$9y^2 \pm 18 + 1 = 0$. Thus $y^2 = \frac{-1\mp 18}{9} = \frac{17}{9}$ and $x = -y$. Thus $y = \frac{\pm\sqrt{17}}{3}$ and $x = \frac{\mp\sqrt{17}}{3}$, and $z = \frac{1}{3}$

Note $d^2 = \frac{17}{9} + \frac{17}{9} + \frac{1}{9} = \frac{35}{9}$

If $y = 0$, then $x = 0$ and $z = 6$, but $f(0, 0, 6) = 36 > \frac{35}{9}$

Answer: $\underline{(-\frac{\sqrt{17}}{3}, \frac{\sqrt{17}}{3}, \frac{1}{3}), (\frac{\sqrt{17}}{3}, -\frac{\sqrt{17}}{3}, \frac{1}{3})}$

Note: you were required to use Lagrange multiplier, but in real life, you could use simpler method for this problem since you can plug in the constraint to obtain a simpler unconstrained optimization problem. But in real life, constraints can be more interesting, and thus one cannot always turn a constrained optimization problem into an unconstrained optimization problem.

Minimizing distance squared $= d^2 = f(x, y, z) = (x-0)^2 + (y-0)^2 + (z-0)^2 = x^2 + y^2 + z^2$

subject to constraint $z = 3xy + 6$

Thus minimizing $h(x, y) = x^2 + y^2 + (3xy + 6)^2$

$\nabla h \;=\; <2x{+}2(3xy{+}6)(3y),\ 2y{+}2(3xy{+}6)(3x)> \;=\; <2x{+}18xy^2{+}36y,\ 2y{+}18x^2y{+}36x> \;=\; <0,\ 0>$

$2x + 18xy^2 + 36y = 0$ and $2y + 18x^2y + 36x = 0$

$2x^2 + 18x^2y^2 + 36yx = 0$ and $2y^2 + 18x^2y^2 + 36xy = 0$

Thus $2x^2 - 2y^2 = 0$. Hence $x = \pm y$

$2y+18x^2y+36x = 0$ implies $2y+18(\pm y)^2y+36(\pm y) = 2y(1\pm 18+9y^2) = 0$. Hence $y = 0$ or $y^2 = \frac{\mp 18-1}{9} = \frac{17}{9}$

Thus the critical points are $(0, 0), (-\frac{\sqrt{17}}{3}, \frac{\sqrt{17}}{3}), (\frac{\sqrt{17}}{3}, -\frac{\sqrt{17}}{3})$

$h(\frac{\mp\sqrt{17}}{3}, \frac{\pm\sqrt{17}}{3}) = \frac{17}{9} + \frac{17}{9} + (-3(\frac{17}{9}) + 6)^2 = \frac{34}{9} + (-\frac{17}{3} + \frac{18}{3})^2 = \frac{34}{9} + (\frac{1}{3})^2 = \frac{34}{9} + \frac{1}{9} = \frac{35}{9}$

$h(0, 0) = 36$

Answer: $\underline{(-\frac{\sqrt{17}}{3}, \frac{\sqrt{17}}{3}, \frac{1}{3}), (\frac{\sqrt{17}}{3}, -\frac{\sqrt{17}}{3}, \frac{1}{3})}$

[12] 6.) Let $R = [0, 3] \times [0, 1]$. Use a partition consisting of 3 unit squares to estimate the volume of the solid under the surface $z = xy + 2y$ and above the rectangle $R = [0, 3] \times [0, 1]$ in the xy-plane. Use the upper right corner of each of the 3 unit squares to estimate the height of the 3 rectangular columns.

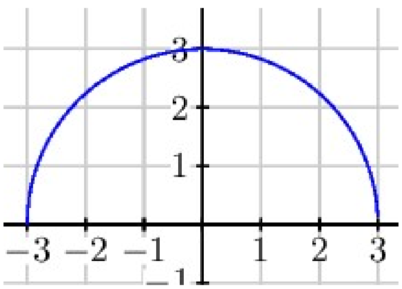
$$f(x_1, y_1)\Delta x \Delta y + f(x_2, y_2)\Delta x \Delta y + f(x_3, y_3)\Delta x \Delta y = f(1, 1)(1)(1) + f(2, 1)(1)(1) + f(3, 1)(1)(1)$$

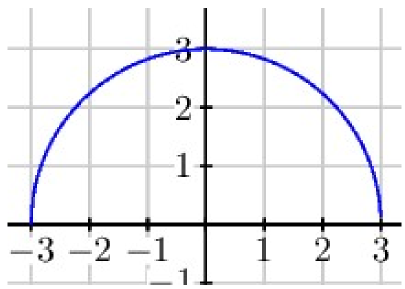
$$= (1 + 2)(1)(1) + (2 + 2)(1)(1) + (3 + 2)(1)(1) = 3 + 4 + 5 = 12$$

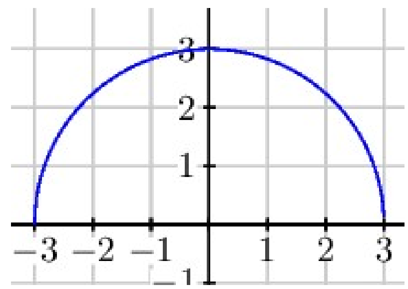
Answer: 12

7.) Write the four double integrals specified below for determining the volume of the solid under the surface $z = \frac{2y}{x^2+y^2}$ and above the region in the xy-plane bounded by the curves $y = 0$ and $y = \sqrt{9 - x^2}$. You do NOT need to evaluate the integral. CIRCLE your answer.

$\frac{2y}{x^2+y^2} = \frac{2r sin\theta}{r^2} = \frac{2sin\theta}{r}$







$0 \le r \le 3, \;\; 0 \le \theta \le \pi$ $\quad$ $-\sqrt{9 - y^2} \le x \le \sqrt{9 - y^2}, \;\; 0 \le y \le 3$ $\quad$ $0 \le y \le \sqrt{9 - x^2}, \;\; -3 \le x \le 3$

[4] 7a.) Use polar coordinates, integrating first with respect to r, and then with respect to θ.

$$\int_0^{\pi} \int_0^3 \frac{2sin\theta}{r} r dr d\theta$$

[4] 7b.) Use polar coordinates, integrating first with respect to θ, and then with respect to r.

$$\int_0^3 \int_0^{\pi} \frac{2sin\theta}{r} r d\theta dr$$

[4] 7c.) Use Euclidean coordinates, integrating first with respect to x, and then with respect to y.

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \frac{2y}{x^2+y^2} dx dy$$

[4] 7d.) Use Euclidean coordinates, integrating first with respect to y, and then with respect to x.

$$\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \frac{2y}{x^2+y^2} dy dx$$