Note calculating $\int_C \mathbf{F} \cdot d\mathbf{r}$ is much easier than calculating $\int\int_S curl\mathbf{F} \cdot d\mathbf{S}$

as we will quickly illustrate on the next 2 slides

$$curl(z^2, -3xy, x^3y^3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & -3xy & x^3y^3 \end{vmatrix}$$

$$= \mathbf{i}\left(\tfrac{\partial(x^3y^3)}{\partial y} - \tfrac{\partial(-3xy)}{\partial z}\right) - \mathbf{j}\left(\tfrac{\partial(x^3y^3)}{\partial x} - \tfrac{\partial(z^2)}{\partial z}\right) + \mathbf{k}\left(\tfrac{\partial(-3xy)}{\partial x} - \tfrac{\partial(z^2)}{\partial y}\right)$$

$$= < 3x^3y^2, -3x^2y^3 + 2z, -3y >$$

$$curl(z^2, -3xy, x^3y^3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & -3xy & x^3y^3 \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial(x^3y^3)}{\partial y} - \frac{\partial(-3xy)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial(x^3y^3)}{\partial x} - \frac{\partial(z^2)}{\partial z}\right) + \mathbf{k}\left(\frac{\partial(-3xy)}{\partial x} - \frac{\partial(z^2)}{\partial y}\right)$$

$$= < 3x^3y^2, -3x^2y^3 + 2z, -3y >$$

On the surface $z = 5 - x^2 - y^2$,

$$curl(z^2, -3xy, x^3y^3) = < 3x^3y^2, -3x^2y^3 + 2(5 - x^2 - y^2), -3y >$$

Recall $d\mathbf{S} = \mathbf{N}dA$ where $\mathbf{N}$ is the normal pointing in the right direction
(up in this case since S is oriented upwards.).

Since our surface is a function of $z = f(x, y) = 5 - x^2 - y^2$, and $\mathbf{N}$ points up.

$$\mathbf{N} = < -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 > = < 2x, 2y, 1 >$$

Let $R =$ circular disk $x^2 + y^2 \leq 4$

$$\int\int_S curl\mathbf{F} \cdot d\mathbf{S}$$

$$= \int\int_R < 3x^3y^2, -3x^2y^3 + 2(5 - x^2 - y^2), -3y > \cdot < 2x, 2y, 1 > dxdy$$

$$\int\int_S curl\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_\Psi curl\mathbf{F} \cdot d\mathbf{\Psi}$$

But we could instead use surface $z = 1$, $x^2 + y^2 \leq 4$

$$\int\int_S curl\mathbf{F} \bullet d\mathbf{S}$$

$$= \int\int_R < 3x^3y^2, -3x^2y^3 + 2(5 - x^2 - y^2), -3y > \bullet < 2x, 2y, 1 > dxdy$$

$$\int\int_\Psi curl\mathbf{F} \bullet d\mathbf{\Psi}$$

$$= \int\int_R < 3x^3y^2, -3x^2y^3 + 2(5 - x^2 - y^2), -3y > \bullet < 0, 0, 1 > dxdy$$

$$= \int\int_R -3ydxdy = \int_0^{2\pi}\int_0^2 -3(rcos\theta)rdrd\theta$$

$$= \int_0^2\int_0^{2\pi} -3r^2cos\theta d\theta dr = \int_0^2 -3r^2sin\theta|_0^{2\pi}dr = \int_0^2 0dr = 0$$

Solve: $y'' + 4 = 0$, $y(0) = -2$, $y'(0) = 0$

$r^2 + 4 = 0$ implies $r^2 = -4$. Thus $r = \pm 2i$.

Since $r = 0 \pm 2i$, $y = c_1cos(2t) + c_2sin(2t)$.

Then $y' = -2c_1sin(2t) + 2c_2cos(2t)$

$y(0) = -2$: $-2 = c_1cos(0) + c_2sin(0)$ implies $-2 = c_1$

$y'(0) = 0$: $0 = -2c_1sin(0) + 2c_2cos(0)$ implies $0 = c_2$

Thus IVP solution: $y = -2cos(2t)$