Note: We will use $|x|$ to denote absolute value of x and $detM$ to denote the determinant of M.

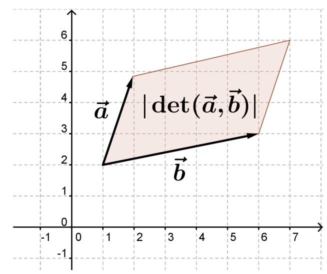
Suppose $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$. Recall that the area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is the length of the vector $\mathbf{a} \times \mathbf{b}$.

Suppose (a_1, a_2) and (b_1, b_2) are vectors in $\mathbb{R}^2$. Note that vectors in $\mathbb{R}^2$ can be embedded in R^3 by adding 0 as the third component.

Thus the area of a parallelogram with sides (a_1, a_2) and (a_1, a_2) is the length of the vector

$$(a_1, a_2, 0) \times (b_1, b_2, 0) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{bmatrix} = (0, 0, a_1 b_2 - b_1 a_2)$$

Thus the length of this vector $= \left| det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right| = |a_1 b_2 - b_1 a_2|$



https://www.webmatematik.dk/lektioner/matematik-b/vektorer-i-2d/determinant

---

Suppose $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in $\mathbb{R}^3$. The scalar triple product of these vectors is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

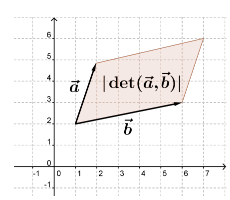
Recall that the volume of the parallelepiped with sides $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the absolute value of the scalar triple product of these vector

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \left| det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right|.$$

This generalizes to any dimension. The n-dimensional volume of an n-dimensional "parallelogram" with sides $\mathbf{a_1}, ..., \mathbf{a_n}$ is the absolute value of the determinant of the matrix with rows $\mathbf{a_i}$, $i = 1, ..., n$.

In particular area of 2 dimensional parallelogram is

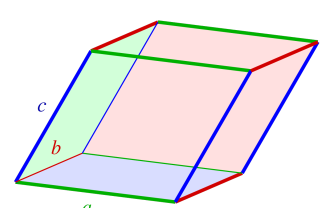
$$\left| det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right|$$



https://www.webmatematik.dk/lektioner/matematik-b/vektorer-i-2d/determinant

And the volume of the 3-dimensional parallelepiped is

$$\left| det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right|$$



https://de.wikipedia.org/wiki/Parallelepiped

We will see in the recommended videos that

In 2 dimensions: $dA = |\frac{\partial(x,y)}{\partial(u,v)}|\ du\ dv$

where $\frac{\partial(x,y)}{\partial(u,v)}$ = Jacobian of $(x(u,v), y(u,v)) = det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \end{bmatrix}$

and $|\frac{\partial(x,y)}{\partial(u,v)}|$ is the absolute value of this Jacobian.

Polar coordinate example: If $(x(r,\theta), y(r,\theta)) = (rcos(\theta), rsin(\theta))$, then

$$\frac{\partial(x,y)}{\partial(r,\theta)} = det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r \quad \text{and} \quad dA = rdrd\theta \qquad \text{(Note: } r \geq 0)$$

Page 1 of these notes implies that this easily generalizes to 3 dimensions:

In 3 dimensions: $dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|\ du\ dv\ dw$

where $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ = Jacobian of $(x(u,v,w), y(u,v,w), z(u,v,w)) = det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$

and $|\frac{\partial(x,y,z)}{\partial(u,v,w)}|$ is the absolute value of this Jacobian.

Cylindrical coordinate example:

If $(x(r,\theta,z), y(r,\theta,z), z(r,\theta,z)) = (rcos(\theta), rsin(\theta), z)$, then

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r \quad \text{and} \quad dV = rdrd\theta dz \qquad \text{(Note: } r \geq 0)$$

Spherical coordinate example:

If $(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi)) = (\rho sin(\phi)cos(\theta), \rho sin(\phi)sin(\theta), \rho cos(\phi))$, then

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = det \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix} = \rho^2 sin(\phi) \quad \text{and} \quad dV = \rho^2 sin(\phi) d\rho d\theta d\phi$$

(Note: $\rho^2 sin(\phi) \geq 0$ since $\rho \geq 0$ and $0 \leq \phi \leq \pi$)

FYI: The above generalizes to higher dimensions.