$\bar{y} = \frac{1}{mass} \iiint_{3D\,region} r\sin\theta\, \delta(r,\theta,z)\, r\,dr\,d\theta\,dz$

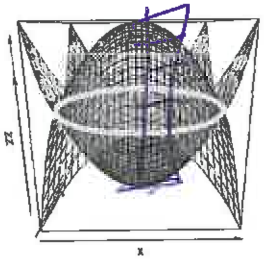
Find the volume between the surfaces: $z = x^2 + y^2 - 9$, $z = 16 - x^2 - y^2$

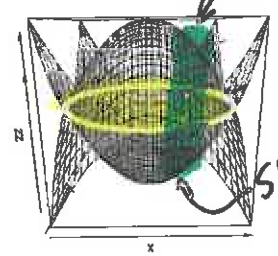
Need to find region over which to integrate:

Find intersection between surfaces: $x^2 + y^2 - 9 = 16 - x^2 - y^2$

Hence intersection is $2x^2 + 2y^2 = 25$

Thus (per figure) integrating over region $x^2 + y^2 \le \frac{25}{2}$





Height of columns: $16 - x^2 - y^2 - (x^2 + y^2 - 9) = 25 - 2x^2 - 2y^2$

Thus need to integrate $\int\int_R (25 - 2x^2 - 2y^2)dA = \iint_R \left[\int dz\right] dA$

Note: this integral is easier to compute using polar coordinates:

Height of columns: $25 - 2x^2 - 2y^2 = 25 - 2r^2$

Region: $x^2 + y^2 \le \frac{25}{2}$ is equivalent to $0 \le r \le \frac{5}{\sqrt{2}}$ and $0 \le \theta \le 2\pi$

Hence $\int\int_R (25 - 2x^2 - 2y^2)(dA) = \boxed{\int_0^{2\pi}\int_0^{\frac{5}{\sqrt{2}}}(25 - 2r^2)(r\,dr\,d\theta)}$

$= \int_0^{\frac{5}{\sqrt{2}}}\int_0^{2\pi}(25r - 2r^3)d\theta dr = \int_0^{\frac{5}{\sqrt{2}}}\int(25r - 2r^3)\theta|_0^{2\pi}dr = \int_0^{\frac{5}{\sqrt{2}}}(25r - 2r^3)(2\pi)dr$

$= (2\pi)(\frac{25}{2}r^2 - \frac{2}{4}r^4)|_0^{\frac{5}{\sqrt{2}}} = (\pi)[25(\frac{5}{\sqrt{2}})^2 - (\frac{5}{\sqrt{2}})^4)] = (\pi)[25(\frac{5^2}{2}) - (\frac{5^4}{4})]$

$= (\pi)[\frac{2(5^4) - 5^4}{4}] = (\pi)[\frac{5^4}{4}] = \frac{625\pi}{4}$

$\delta(r,\theta,z) = r\cos\theta + r\sin\theta + z + 1$

$\delta(x,y,z) = x + y + z + 1$

$z = r^2 - 9, \quad z = 16 - r^2$

Find the volume between the surfaces: $z = x^2 + y^2 - 9$, $z = 16 - x^2 - y^2$

Use a triple integral:

$\int_0^{2\pi}\int_0^{\frac{5}{\sqrt{2}}}\left[\int_{r^2-9}^{16-r^2} dz\right] r\,dr\,d\theta$

If the density of this volume is $\delta(x, y, z) = x + y + z + 1$, find the mass of this volume.

$\int_0^{2\pi}\int_0^{\frac{5}{\sqrt{2}}}\int_{r^2-9}^{16-r^2}\left[r\cos\theta + r\sin\theta + z + 1\right] \underbrace{r\,dz\,dr\,d\theta}_{dV}$

$r\,dz\,dr\,d\theta$

The centroid is

$\bar{x} =$

$\bar{x} = \frac{1}{mass}\int_0^{2\pi}\int_0^{\frac{5}{\sqrt{2}}}\int_{r^2-9}^{16-r^2}(r\cos\theta)\left[r\cos\theta + r\sin\theta + z + 1\right] r\,dz\,dr\,d\theta$

$\bar{y} =$

$\bar{y} = \frac{1}{mass}\iiint_{3d\,Region} y\,\delta(x,y,z)\,dV$

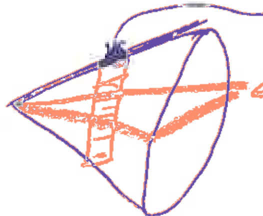
$y = r\sin\theta$ in cylindrical cood

$\bar{z} =$

$\bar{z} = \frac{1}{mass}\iiint z\,\delta(r,\theta,z)\,dV$

Euclid & cylindrical

$y=3$



$z=f(x,y)$

largest cross section occurs when $z=0 \Rightarrow y^2=x^2$

Find the volume of the region bounded by $y^2 = x^2 + z^2$ and $y = 3$ using a triple integral.

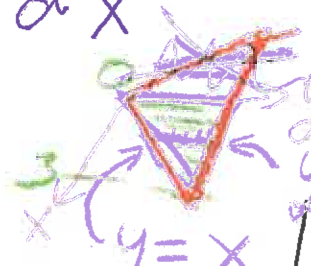
$z^2 = y^2 - x^2$

Using Euclidean coordinates:

$z = \pm\sqrt{y^2-x^2}$

Integrate first with respect to z, then y, then x

$$2\int_0^3 \int_x^3 \int_{-\sqrt{y^2-x^2}}^{\sqrt{y^2-x^2}} dz\, dy\, dx$$

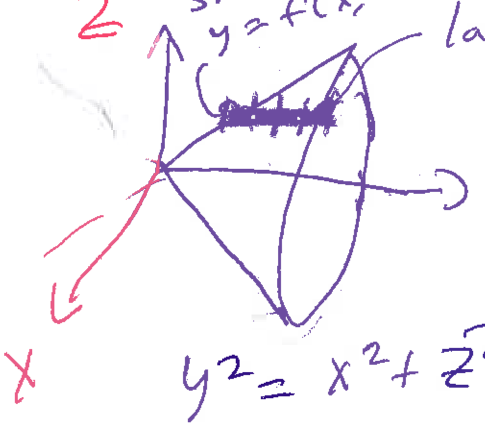


3 $y=3$ $y=x$

Integrate first with respect to y, then z, then x

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+z^2}}^{3} dy\, dz\, dx$$

largest cross-section is when $y=3$: $x^2+z^2=9$

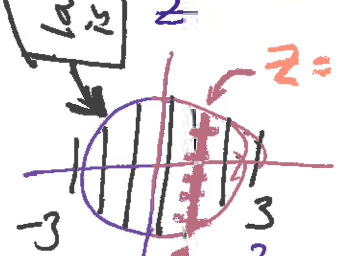


smaller $y=f(x,z)=\sqrt{x^2+z^2}$

larger $y=f(x,z)=3$

z y x

$y^2 = x^2+z^2 \Rightarrow y = \pm\sqrt{x^2+z^2}$



z x -3 3

$z=\sqrt{9-x^2}$

$-\sqrt{9-x^2}$

$x^2+z^2=3^2$

$z=\pm\sqrt{9-x^2}$

Note the volume of the region bounded by $y^2 = x^2 + z^2$ and $y = 3$ is the same as the volume of the region bounded by $z^2 = x^2 + y^2$ and $z = 3$

$z^2 = r^2 \Rightarrow z = \pm r \leftarrow$ but want $+z=r$

Use Cylindrical coordinates to find the volume of the region bounded by $z^2 = x^2 + y^2$ and $z = 3$.

$z=3$

$z=\pm r$

Integrate first with respect to r, then θ, then z.

$z=\pm r$

$z^2=x^2+y^2$

$$\int_0^3 \int_0^{2\pi} \int_0^{z} \underbrace{r\, dr\, d\theta\, dz}_{dV}$$

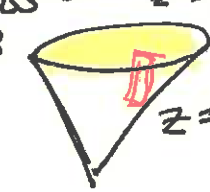
adding up circular disks

Integrate first with respect to z, then r, then θ

$$\int_0^{2\pi} \int_0^3 \left[\int_r^3 dz\right] r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 r \left[\int_r^3 dz\right] dr\, d\theta$$

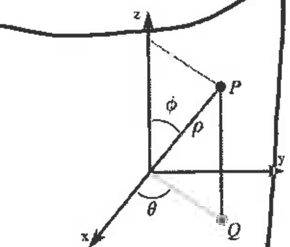
largest cross-section $z=3$ when $z=3$



$z=r$

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

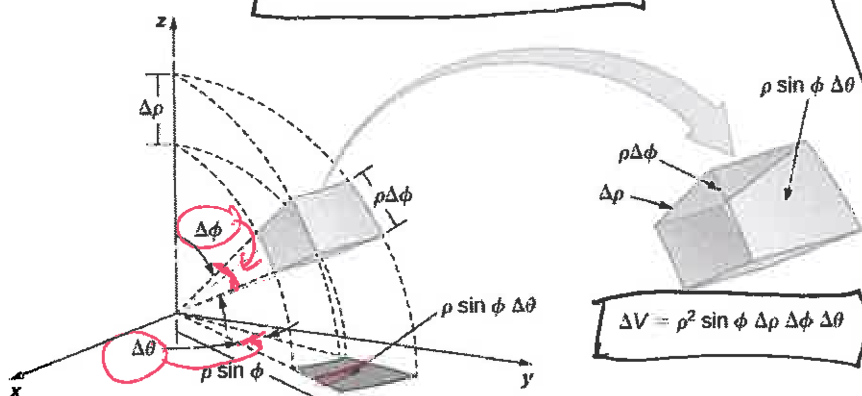
Spherical coordinates:



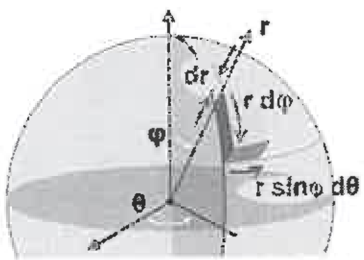
$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta \qquad (1)$$
$$z = \rho \cos\phi.$$

$$\rho \geq 0$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \varphi \leq \pi$$

https://mathinsight.org/spherical_coordinates



$$\Delta V = \rho^2 \sin\phi \, \Delta\rho \, \Delta\phi \, \Delta\theta$$

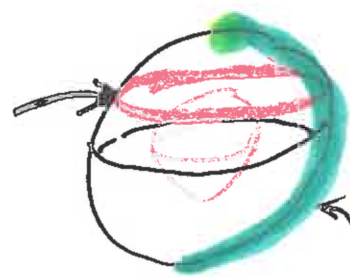


Use spherical coordinates to find the volume between spheres of radius 3 and radius 4.

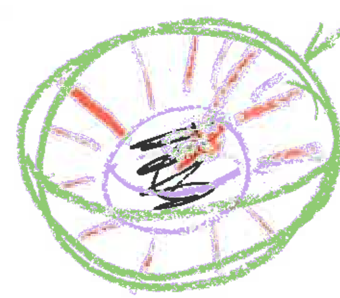
$$\iiint dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_3^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$0 \leq \theta \leq 2\pi$



$0 \leq \varphi \leq \pi$



$\rho = 4$

$\rho = 3$