12.7: Chain rule

Note: If you have more than one independent variable and more than one dependent variable at any stage of the problem, then you must use Jacobian matrices (for example HW 12.7: 5, 13).

Let $g(x,y) = x^2 + y^2$

Let $T(r,\theta) = (r cos\theta, r\theta) = (x(r,\theta), y(r,\theta))$

Then $(g \circ T)(r,\theta) = g(T(r,\theta)) = g(r cos\theta, r sin\theta) = r^2cos^2\theta + r^2 sin^2\theta = r^2$

Thus $(g \circ T)'(r,\theta) = [\frac{\partial(g\circ T)}{\partial r} \quad \frac{\partial(g\circ T)}{\partial \theta}] = [\frac{\partial r^2}{\partial r} \quad \frac{\partial r^2}{\partial \theta}] = [2r \quad 0]$

$\boxed{\text{Chain rule:} \quad (g \circ T)'(r,\theta) = g'(T(r,\theta))T'(r,\theta)}$

Note that $g'(T(r,\theta)) = [\frac{\partial(x^2+y^2)}{\partial x} \quad \frac{\partial(x^2+y^2)}{\partial x}] = [2x \quad 2y] = [2r cos\theta \quad 2r sin\theta]$

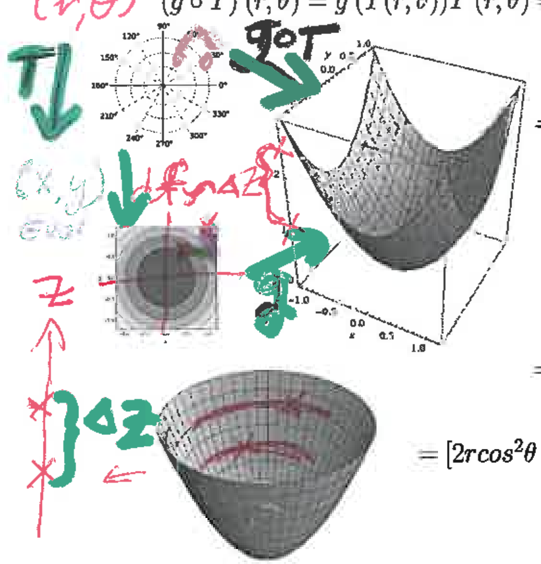
where x and y are functions of r and θ

That is $x = r cos\theta$ and $y = r\theta$

and $g'(x,y)$ is evaluated at $T(r,\theta)$.

polar coordinate (r,θ)

$$(g \circ T)'(r,\theta) = g'(T(r,\theta))T'(r,\theta) = [\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial x}]\begin{bmatrix} \frac{\partial x(r,\theta)}{\partial r} & \frac{\partial x(r,\theta)}{\partial \theta} \\ \frac{\partial y(r,\theta)}{\partial r} & \frac{\partial y(r,\theta)}{\partial \theta} \end{bmatrix}$$



$$= [\frac{\partial(x^2+y^2)}{\partial x} \quad \frac{\partial(x^2+y^2)}{\partial x}]\begin{bmatrix} \frac{\partial x(r,\theta)}{\partial r} & \frac{\partial x(r,\theta)}{\partial \theta} \\ \frac{\partial y(r,\theta)}{\partial r} & \frac{\partial y(r,\theta)}{\partial \theta} \end{bmatrix}$$

$$= [2x \quad 2y]\begin{bmatrix} \frac{\partial(r cos\theta)}{\partial r} & \frac{\partial(r cos\theta)}{\partial \theta} \\ \frac{\partial(r sin\theta)}{\partial r} & \frac{\partial(r sin\theta)}{\partial \theta} \end{bmatrix}$$

$$= [2r cos\theta \quad 2r sin\theta]\begin{bmatrix} cos\theta & -r sin\theta \\ sin\theta & r cos\theta \end{bmatrix}$$

$$= [2r cos^2\theta + 2r sin^2\theta \quad -2r^2 cos\theta sin\theta + 2r^2 sin\theta cos\theta]$$

$$= [2r \quad 0]$$

If $T(r,\theta) = (x(r,\theta), y(r,\theta))$ and if $z = g(x,y)$, $T: \mathbb{R}^2 \to \mathbb{R}^2$

Then $z = (g \circ T)(r,\theta)$ $g: \mathbb{R}^2 \to \mathbb{R}$

1×2 $\quad$ 2×2

and $[\frac{\partial z}{\partial r} \quad \frac{\partial z}{\partial \theta}] = (g \circ T)'(r,\theta) = g'(T(r,\theta))T'(t,\theta) = [\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y}]\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$

$$= [\frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}]$$

Thus $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$ and $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$

This matches with 12.6 were we saw $\Delta z \sim dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

Application: Suppose that sand is falling at a rate of 40 πm^3/sec, forming a conical pile. If the height of the pile is increasing at a rate of 0.5 m/sec when the height of the pile is 6m and its radius is 12m, find the rate at which the radius is increasing $\quad$ indep variable = time

Note volume of cone is given by $V = \pi r^2(\frac{h}{3})$

Implicit diff so don't need matrices

Example: If z is a function of x and y and if $x ln|z| + y sin(z) = xyz + x$, find $\frac{\partial z}{\partial x}$.

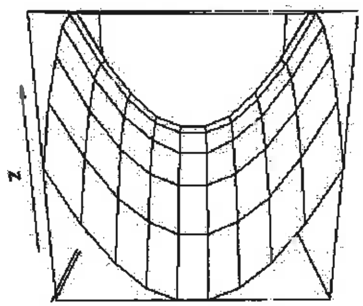
Note z is defined implicitly. We will discuss implicit function theorem more thoroughly later this semester.

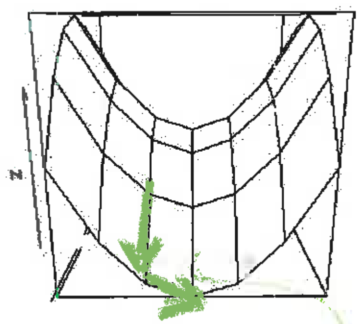
Recall the vector $(1, 0, \frac{\partial z}{\partial x})$ is tangent to the surface defined by the above equation.

$$\left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right)$$

Section 12.6: Math 5 (multiple independent variables)

Suppose $z = f(x,y)$. Then the gradient vector of f is: $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$





Note $\Delta z \sim df = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \cdot (\Delta x, \Delta y) = \nabla f \cdot (\Delta x, \Delta y)$

change wrt z ~ change wrt x + chang wrt y

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$[(x_0 + \Delta x,\ y_0,\ z_0 + \frac{\partial f}{\partial x}\Delta x) - (x_0, y_0, z_0)]$

$+[(x_0,\ y_0 + \Delta y,\ z_0 + \frac{\partial f}{\partial y}\Delta y) - (x_0, y_0, z_0)]$

$= [(\Delta x,\ 0,\ \frac{\partial f}{\partial x}\Delta x)] + [(0,\ \Delta y,\ \frac{\partial f}{\partial y}\Delta y)] = (\Delta x,\ \Delta y,\ \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y)$

$= (\Delta x,\ \Delta y,\ \nabla f \cdot (\Delta x, \Delta y)) \sim (\Delta x,\ \Delta y,\ \Delta z)$

Approximate $(10^{\frac{4}{3}})(24^{\frac{1}{2}})$.

Let $f(x,y) = (x^{\frac{4}{3}})(y^{\frac{1}{2}})$. Use $f(8,25)$ to approximate $f(10,24)$

Note x and y are independent variables, so

$(dx, dy) = (\Delta x, \Delta y) = (10-8, 24-25) = (2,-1)$

$f(8,25) = (8^{\frac{4}{3}})(25^{\frac{1}{2}}) = (2^4)(5) = 16(5) = 80$

$\frac{\partial f}{\partial x}(x,y) = \frac{4}{3}x^{\frac{1}{3}}(y^{\frac{1}{2}})$ and $\frac{\partial f}{\partial y}(x,y) = (x^{\frac{4}{3}})\frac{1}{2}y^{-\frac{1}{2}}$

$\frac{\partial f}{\partial x}(8,25) = \frac{4}{3}(8^{\frac{1}{3}})(25^{\frac{1}{2}}) = \frac{4}{3}(2)(5) = \frac{40}{3}$ and

$\frac{\partial f}{\partial y}(8,25) = (8^{\frac{4}{3}})(\frac{1}{2})(25^{-\frac{1}{2}}) = (16)(\frac{1}{2})(\frac{1}{5}) = \frac{8}{5}$ Thus $\nabla f(8,25) = (\frac{40}{3}, \frac{8}{5})$

$(10^{\frac{4}{3}})(24^{\frac{1}{2}}) = f(10,24) = f(8,25) + [f(10,24) - f(8,25)] = f(8,25) + \Delta f$

$\sim f(8,25) + df = f(8,25) + \nabla f \cdot (\Delta x, \Delta y) = 80 + (\frac{40}{3}, \frac{8}{5}) \cdot (2,-1)$

$= 80 + \frac{80}{3} - \frac{8}{5} = \frac{80(15)}{15} + \frac{400}{15} - \frac{24}{15} = \frac{1200}{15} + \frac{376}{15} = \frac{1576}{15}$

Calc 1: Let $z = f(x)$ and $h = \Delta x = dx$, then

$f(a+h) = f(a) + \Delta z \sim f(a) + df = f(a) + f'(a)h$

Thus $f(a+h) - f(a) \sim f'(a)h$. Hence $\frac{f(a+h)-f(a)}{h} \sim f'(a)$

I.e, $\frac{f(a+h)-f(a)-f'(a)h}{h} \sim 0$. I.e, $\lim_{h\to 0} \frac{f(a+h)-f(a)-f'(a)h}{h} = 0$

Multivariable Calc: Let $z = f(x,y)$, $\mathbf{a} = (x_0, y_0)$ and $\mathbf{h} = (\Delta x, \Delta y)$.

Then $f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \Delta z \sim f(\mathbf{a}) + df = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h}$

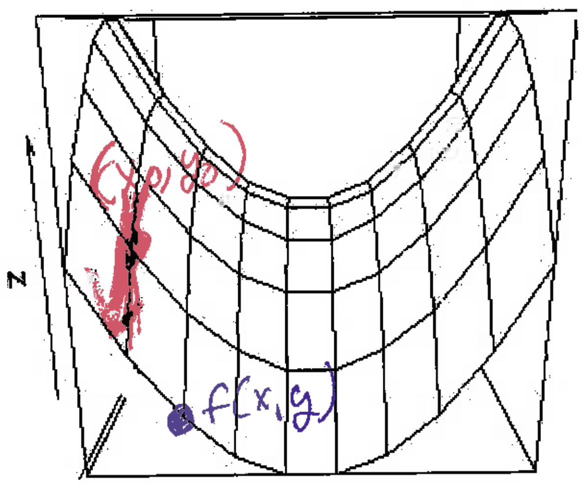
Hence $\lim_{h\to 0} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a})\cdot\mathbf{h}}{|\mathbf{h}|} = 0$

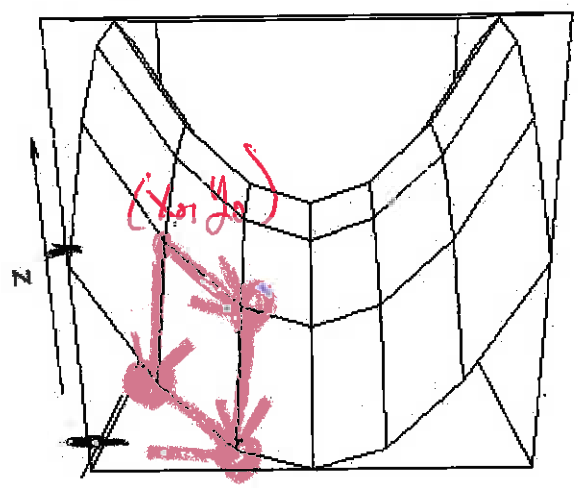
Defn: f is differentiable at $\mathbf{a}$ if these exists a constant vector $\mathbf{c}$ such that

$\lim_{h\to 0} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c}\cdot\mathbf{h}}{|\mathbf{h}|} = 0$ Note $\mathbf{c} = \nabla f(\mathbf{a})$

Section 12.6: Math 5 (multiple independent variables)

Suppose $z = f(x,y)$. Then the gradient vector of f is: $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$





Note $\Delta z \sim df = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \cdot (\Delta x, \Delta y) = \nabla f \cdot (\Delta x, \Delta y)$

total change in z = (change in z in x direction) + (Change in z in y-directi

(x_0, y_0)

$(x_0, y_0 + \Delta y, z_0 + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y)$

$(x_0 + \Delta x, y_0, z_0 + \frac{\partial f}{\partial x}(x_0, y_0)\Delta x)$

$$[(x_0 + \Delta x,\ y_0,\ z_0 + \tfrac{\partial f}{\partial x}\Delta x) - (x_0, y_0, z_0)]$$
$$+[(x_0,\ y_0 + \Delta y,\ z_0 + \tfrac{\partial f}{\partial y}\Delta y) - (x_0, y_0, z_0)]$$

$$= [(\Delta x,\ 0,\ \tfrac{\partial f}{\partial x}\Delta x)] + [(0,\ \Delta y,\ \tfrac{\partial f}{\partial y}\Delta y)] = (\Delta x,\ \Delta y,\ \tfrac{\partial f}{\partial x}\Delta x + \tfrac{\partial f}{\partial y}\Delta y)$$

$$= (\Delta x,\ \Delta y,\ \nabla f \cdot (\Delta x, \Delta y)) \sim (\Delta x,\ \Delta y,\ \Delta z)$$