$K(x) = \left| \frac{2}{(\sqrt{1+4x^2})^3} \right|$

$K'(x) = 0$ to find max/min

11.6: Curvature

Arclength $s(t) = \int_a^t v(t)dt$ where $v(t) = |\mathbf{v}(t)|$

$s(t)$ is an increasing function and thus $s^{-1}(t)$ exists. Let $t(s) =^{-1} (t)$.

Arc-length parametrization = reparametrize by replacing t with $t(s)$.

Example: $r(t) = (cos(t), sin(t), t)$

Unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{velocity}{speed} = \frac{\frac{dr}{dt}}{\frac{ds}{dt}} = \frac{dr}{ds}$ unit $\Rightarrow$ length $= 1$

Thus if $\mathbf{T}$ is parametrized by arclength s, then

$$\mathbf{T}(s) = \frac{\mathbf{v}(s)}{|\mathbf{v}(s)|} = \frac{velocity}{speed} = \frac{\frac{dr}{ds}}{\frac{ds}{ds}} = \frac{dr}{ds}$$

speed = 1 w/ arclength parametrization

Since $\mathbf{T}$ is a unit vector, $\mathbf{T} \cdot \mathbf{T} = 1$

Differentiate with respect to s: $2\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} = 0$

Thus $\mathbf{T}$ is perpendicular to $\frac{d\mathbf{T}}{ds}$

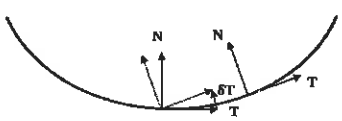
$\vec{T} \cdot \vec{T} = \sum T_i^2$

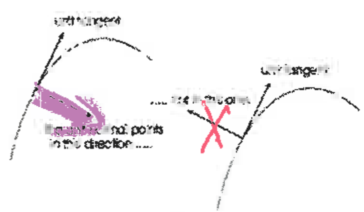
$\frac{d}{ds}(\vec{T} \cdot \vec{T}) = \sum 2T_i \cdot \frac{dT_i}{ds} = 2\vec{T} \cdot \frac{d\vec{T}}{ds}$

Definition: The **principal unit normal** is $\frac{\frac{d\mathbf{T}}{ds}}{|\frac{d\mathbf{T}}{ds}|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$ where

$$\textbf{curvature } \kappa = |\frac{d\mathbf{T}}{ds}| = |\frac{d\mathbf{T}}{dt}\frac{dt}{ds}| = \frac{1}{\frac{ds}{dt}}|\frac{d\mathbf{T}}{dt}| = \frac{1}{speed}|\frac{d\mathbf{T}}{dt}|$$

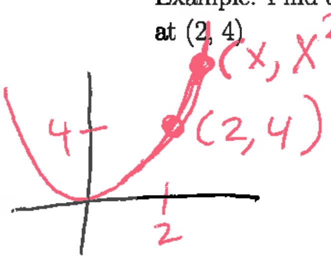
The unit normal points in the direction in which the curve is curving:





https://en.wikipedia.org/wiki/Curvature, http://sites.millersville.edu/bikenaga/calculus/tangent-normal-curvature/tangent-normal-curvature.html

Example: Find the unit tangent and normal vectors to the curve $y = x^2$ at $(2, 4)$



$r(x) = (x, x^2)$

$r(t) = (t, t^2)$

not using arc length paramth

$K = \left| \frac{y''}{|(x', y')|^3} \right|$

In 2D, if $r(t) = (x(t), y(t))$, let $\phi = tan^{-1}(\frac{y'(t)}{x'(t)})$

Write unit tangent in polar coordinates: $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{i}cos\phi + \mathbf{j}sin\phi$

Then $\frac{d\mathbf{T}}{ds} = (-\mathbf{i}sin\phi + \mathbf{j}cos\phi)\frac{d\phi}{ds}$ is obviously perpendicular to $\mathbf{T}$.

curvature $= \kappa = |\frac{d\mathbf{T}}{ds}| = |(-\mathbf{i}sin\phi + \mathbf{j}cos\phi)\frac{d\phi}{ds}| = |\frac{d\phi}{ds}|$

Since $\phi = tan^{-1}(\frac{y'(t)}{x'(t)})$, then

If $r(x) = (x, f(x)) = (x, y)$

$$\boxed{\text{curvature} = \kappa = |\frac{d\phi}{ds}| = \frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{\frac{3}{2}}}}$$

$\Rightarrow x' = 1, \; x'' = 0$

Example: Find the point(s) on the curve $y = x^2$ where curvature is maximum.

$y' = 2x$

$y'' = 2$

Maximize $K(x) = \left| \frac{2}{|(1, 2x)|^3} \right|$

$\vec{v}(t) = \vec{r}'(t) = (1, 2t)$

$\vec{T}(t) = \frac{(1, 2t)}{\sqrt{1+4t^2}}$

At $(2, 4)$, $\vec{T}(2) = \frac{(1, 4)}{\sqrt{17}} = \left(\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$

$\vec{N} = \left(\frac{-4}{\sqrt{17}}, \frac{1}{\sqrt{17}}\right)$

$y = x^2 \Rightarrow y' = 2x$

slope is $\frac{4}{1}$

$r(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right)$

11.6: Curvature

Arclength $s(t) = \int_a^t v(t)dt$ where $v(t) = |\mathbf{v}(t)|$

$s(t)$ is an increasing function and thus $s^{-1}(t)$ exists. Let $t(s) =^{-1}(t)$.

Arc-length parametrization = reparametrize by replacing t with $t(s)$.

Example: $r(t) = (cos(t), sin(t), t)$

Unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{velocity}{speed} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{d\mathbf{r}}{ds}$

Thus if $\mathbf{T}$ is parametrized by arclength s, then

$$\mathbf{T}(s) = \frac{\mathbf{v}(s)}{|\mathbf{v}(s)|} = \frac{velocity}{speed} = \frac{\frac{d\mathbf{r}}{ds}}{\frac{ds}{ds}} = \frac{d\mathbf{r}}{ds}$$

if r parametrized by s $|r'(s)| = 1$

Since $\mathbf{T}$ is a unit vector, $\mathbf{T} \cdot \mathbf{T} = 1. = \sum T_i^2$

Differentiate with respect to s: $2\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} = 0$

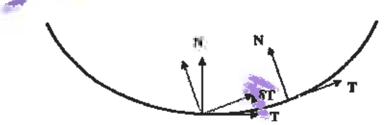
Thus $\mathbf{T}$ is perpendicular to $\frac{d\mathbf{T}}{ds}$

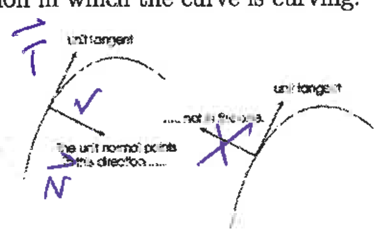
Definition: The **principal unit normal** is $\frac{\frac{d\mathbf{T}}{ds}}{|\frac{d\mathbf{T}}{ds}|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$ where

$$\textbf{curvature}\ \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| = \frac{1}{\frac{ds}{dt}}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{speed}\left|\frac{d\mathbf{T}}{dt}\right|$$

The unit normal points in the direction in which the curve is curving:

ΔT





https://en.wikipedia.org/wiki/Curvature, http://sites.millersville.edu/bikenaga/calculus/tangent-normal-curvature/tangent-normal-curvature.html

In 2D, if $r(t) = (x(t), y(t))$, let $\phi = tan^{-1}(\frac{y'(t)}{x'(t)})$

Write unit tangent in polar coordinates: $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{i}cos\phi + \mathbf{j}sin\phi$

Then $\frac{d\mathbf{T}}{ds} = (-\mathbf{i}sin\phi + \mathbf{j}cos\phi)\frac{d\phi}{ds}$ is obviously perpendicular to $\mathbf{T}$.

curvature $= \kappa = |\frac{d\mathbf{T}}{ds}| = |(-\mathbf{i}sin\phi + \mathbf{j}cos\phi)\frac{d\phi}{ds}| = |\frac{d\phi}{ds}|$

Since $\phi = tan^{-1}(\frac{y'(t)}{x'(t)})$, then

curvature $= \kappa = |\frac{d\phi}{ds}| = \frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{\frac{3}{2}}}$

$$\left|\frac{d\phi}{ds}\right| = \left|\frac{y''}{|(x', y')|^2}\right|$$

If $r(x) = (x, y)$

$x' = 1, \quad x'' = 0$

$y' = 2x$

$y'' = 2$

Example: Find the point(s) on the curve $y = x^2$ where curvature is maximum.

$$K(x) = \left|\frac{2}{\sqrt{1^2 + (2x)^2}^2}\right|$$

use calc 1 to find max $K'(x) = 0$

unit tangent at $(2, 4)$

$\vec{T}(2) = \left(\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$ since $\underbrace{\sqrt{1 + 4(2)^2}}_{\text{length}}$

Note $(x^2)' = 2x$

slope at $(2, 4) = 4$

direction $(1, 4)$

run rise

$\vec{N}(2) = \left(\frac{-4}{\sqrt{14}}, \frac{1}{\sqrt{17}}\right)$

Example: Find the unit tangent and normal vectors to the curve $y = x^2$ at $(2, 4)$

equiv $\begin{cases} r(x) = (x, x^2) \\ r(t) = (t, t^2) \end{cases}$

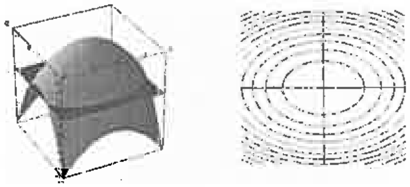
not using arclength

~~$|\vec{r}(x)| = \sqrt{1^2 + (2x)^2}$~~

$\vec{V}(x) = (1, 2x) \Rightarrow \vec{T}(x) = \frac{(1, 2x)}{\sqrt{1 + 4x^2}}$

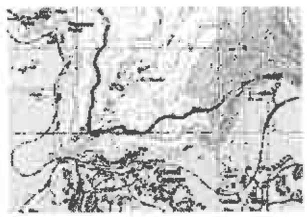
12.2 Functions of several variables: $z = f(x_1, ..., x_n)$

Level set: $f(x_1, ..., x_n) = c$ for some constant c.



Nykamp DQ, "Level sets." From Math Insight. http://mathinsight.org/level_sets

A topographical map shows level sets:



https://www.math.tamu.edu/~mpilant/math696/m696_240/jsamayoa/public_html/levelcurves.html

Can use level sets to understand graphs of functions with 3 variables:

Example: $f(x, y, x) = x^2 + y^2 + z^2$

$= x^2 + y^2 + z^2$

$C = x^2 + y^2 + z^2$ } level sets are spheres centered at $(0,0,0)$

Example: $g(x, y, x) = x^2 + y^2 + z^2 - 8z$

$C = x^2 + y^2 + (z-4)^2 - 16$

levelsets = spheres centered at $(0,0,4)$

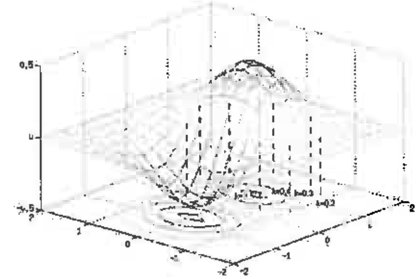
Cross section = the intersection of the graph of $z = f(x, y)$ with a plane.

Examples:

$z = f(c, y)$

$z = f(x, c)$

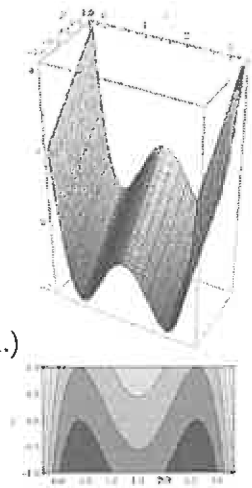
$c = f(x, y)$



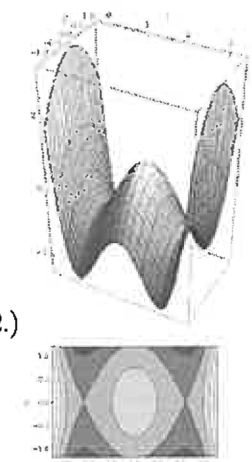
https://www.math.tamu.edu/~mpilant/math696/m696_240/jsamayoa/public_html/levelcurves.html

Match the following graphs to one of the functions below:

1.)



2.)



https://www.wolframalpha.com/

a.) $f(x, y, z) = 6x^3 + 11x^2 - 6x + y$

b.) $f(x, y, z) = x^4 - 6x^3 + 11x^2 - 6x + y$

c.) $f(x, y, z) = x^4 - 6x^3 + 11x^2 - 6x - y^2$

d.) $f(x, y, z) = x^4 - 6x^3 + 11x^2 - 6x + y^2$