12.7: 5

Suppose $w = ln(x^2 + y^2 + z^2)$, $\quad x = s - t$, $\;y = s + t$, $\;z = 2\sqrt{st} = 2(st)^{\frac{1}{2}}$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$.

Short answer:

$$\begin{bmatrix} \frac{\partial w}{\partial s} & \frac{\partial w}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{2x}{x^2+y^2+z^2} & \frac{2y}{x^2+y^2+z^2} & \frac{2z}{x^2+y^2+z^2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ s^{-\frac{1}{2}}t^{\frac{1}{2}} & s^{\frac{1}{2}}t^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \frac{2x+2y+2zs^{-\frac{1}{2}}t^{\frac{1}{2}}}{x^2+y^2+z^2} & \frac{-2x+2y+2zs^{\frac{1}{2}}t^{-\frac{1}{2}}}{x^2+y^2+z^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2(s-t)+2(s+t)+2(2\sqrt{st})s^{-\frac{1}{2}}t^{\frac{1}{2}}}{(s-t)^2+(s+t)^2+(2\sqrt{st})^2} & \frac{-2(s-t)+2(s+t)+2(2\sqrt{st})s^{\frac{1}{2}}t^{-\frac{1}{2}}}{s^2-2st+t^2+s^2+2st+t^2+4st} \end{bmatrix} = \begin{bmatrix} \frac{2(s-t+s+t+2t)}{2s^2+4st+2t^2} & \frac{2(-s+t+s+t+2s)}{2(s+t)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2s+2t}{(s+t)^2} & \frac{2t+2s}{(s+t)^2} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+t} & \frac{2}{s+t} \end{bmatrix}$$

Note at some point, one needs to evaluate F' at $T(t, s) = (x(s, t), y(s, t), z(s, t)) = (s - t, s + t, 2s^{\frac{1}{2}}t^{\frac{1}{2}})$. When you do so is up to personal preference.

NOTE: If you are asked for $\frac{\partial w}{\partial s}$, then x, y, z should not appear in your answer.

Longer answer: Let $w = F(x, y, z) = ln(x^2+y^2+z^2)$ and $T(s, t) = (x(s, t), y(s, t), z(s, t)) = (s - t, s + t, 2s^{\frac{1}{2}}t^{\frac{1}{2}})$

Note x, y, z are independent variables with respect to the function F. But for the function T, x, y, z are dependent variables (depending on the variables s, t). Note that we often use dependent variables as function names: for example, the function $x(s, t) = s - t$ instead of just writing the variable $x = s - t$

To calculate $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$, we note $w = (F \circ T)(s, t) = F(T(s, t))$.

Since $(F \circ T)(s, t)$ is a single function of two variables,$(F \circ T)'(s, t)$ is the 1×2 matrix $\begin{bmatrix} \frac{\partial w}{\partial s} & \frac{\partial w}{\partial t} \end{bmatrix}$.

Note the derivative of a multivariable function is the Jacobian matrix. By the chain rule,

$$\begin{bmatrix} \frac{\partial w}{\partial s} & \frac{\partial w}{\partial t} \end{bmatrix} = (F \circ T)(s, t) = F'(T(s, t))T'(s, t) = \begin{bmatrix} \frac{2x}{x^2+y^2+z^2} & \frac{2y}{x^2+y^2+z^2} & \frac{2z}{x^2+y^2+z^2} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2(s-t)}{(s-t)^2+(s+t)^2+(2\sqrt{st})^2} & \frac{2(s+t)}{(s-t)^2+(s+t)^2+(2\sqrt{st})^2} & \frac{2(2\sqrt{st})}{(s-t)^2+(s+t)^2+(2\sqrt{st})^2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ s^{-\frac{1}{2}}t^{\frac{1}{2}} & s^{\frac{1}{2}}t^{-\frac{1}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2(s-t)}{s^2-2st+t^2+s^2+2st+t^2+4st} & \frac{2(s+t)}{2s^2+2t^2+4st} & \frac{4\sqrt{st}}{2(s^2+2st+t^2)} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ s^{-\frac{1}{2}}t^{\frac{1}{2}} & s^{\frac{1}{2}}t^{-\frac{1}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s-t}{s^2+2st+t^2} & \frac{s+t}{s^2+2st+t^2} & \frac{2\sqrt{st}}{s^2+2st+t^2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ s^{-\frac{1}{2}}t^{\frac{1}{2}} & s^{\frac{1}{2}}t^{-\frac{1}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s-t+s+t+(2\sqrt{st})s^{-\frac{1}{2}}t^{\frac{1}{2}}}{(s+t)^2} & \frac{-s+t+s+t+(2\sqrt{st})s^{\frac{1}{2}}t^{-\frac{1}{2}}}{(s+t)^2} \end{bmatrix} = \begin{bmatrix} \frac{2s+2t}{(s+t)^2} & \frac{2t+2s}{(s+t)^2} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+t} & \frac{2}{s+t} \end{bmatrix}$$