12.9 Lagrange Multiplier Thm to solve constrained optimization problem:

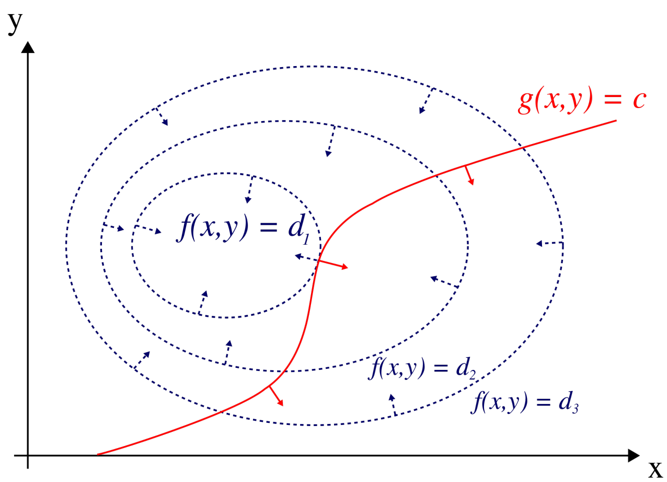
If a maximum or minimum of

$$z = f(x, y) \text{ subject to the constraint } g(x, y) = 0$$

occurs at point $\mathbf{p}$, then either

1.) $\nabla g(\mathbf{p}) = 0$

2.) There exists a constant λ such that $\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p})$



https://en.wikipedia.org/wiki/Lagrange_multiplier

Thus to find extrema for constrained optimization problem,

1.) Find all $\mathbf{p}$ that satisfy $\nabla g(\mathbf{p}) = 0$.

2.) Find all $\mathbf{p}$ that satisfy $\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p})$ for some constant λ.

3.) Check if the points found in steps 1 and 2 satisfy the constraint $g(x, y) = 0$

Example: Find maximum $z = -x^2 - y^2 + 25$ subject to the constraint $xy - 1 = 0$.

Note $f(x, y) = -x^2 - y^2 + 25$ and $g(x, y) = xy - 1 = 0$

Easier method 1: Since one can solve $g(x, y) = 0$ for y (or x), one can use constraint to solve for y (or x) and plug into f, and then use (1) algebra or (2) calc 1 or (3) section 12.5

Method 2: Lagrange multiplier

Note: normally only use Lagrange multiplier when can't (or don't want to) solve constraint for y (or x).

$\nabla g =< y, x >$

If $\nabla g =< y, x >=< 0, 0 >$, then $(x, y) = (0, 0)$, but $g(0, 0) = 0 - 1 \neq 0$. Thus (0, 0) does not satisfy the constraint.

$\nabla f =< -2x, -2y >$

Solve $< -2x, -2y >= \lambda < y, x >$

$-2x \;=\; \lambda y$ and $-2y \;=\; \lambda x$

Thus $-2x^2 \;=\; \lambda xy = -2y^2$

Thus $x^2 = y^2$

Thus $(x, y) = (1, 1), (1, -1), (-1, 1), (-1, -1)$

BUT only 2 of these points satisfy the constraint $g(x, y) = xy - 1 = 0$

$g(1, 1) = 1 - 1 = 0$ $\qquad$ $g(1, -1) = -1 - 1 \neq 0$

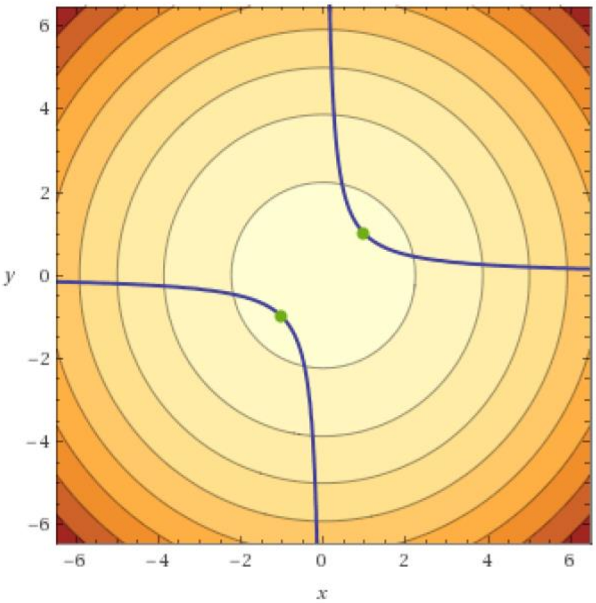
$g(-1, -1) = 1 - 1 = 0$ $\qquad$ $g(-1, 1) = -1 - 1 \neq 0$

Thus only (1, 1) and (-1, -1) satisfy the constraint.

Thus maximum subject to constraint occurs at (1, 1) and (-1, -1).

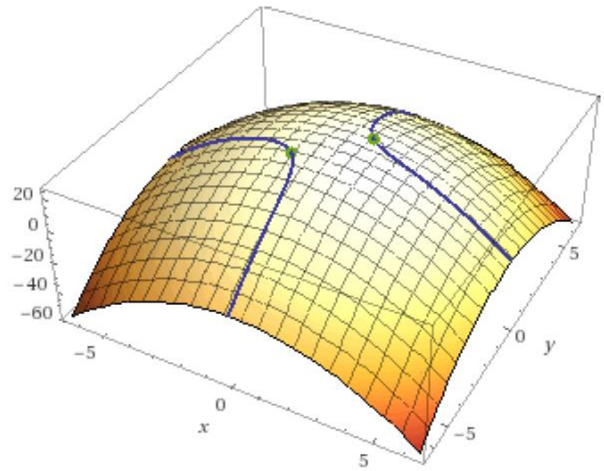
The maximum is $f(x, y) = -(\pm 1)^2 - (\pm 1)^2 + 25 = 23$

Contour plot:



WolframAlpha

minimize -x^2-y^2 + 25 on xy = 1



12.10: 2nd order derivative test

Suppose $z = f(x, y)$

Recall the derivative matrix of f is $Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$

Hessian matrix =

$$D^2 f = \begin{bmatrix} \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) & \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) \\ \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) & \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = Hf$$

Determinant of the Hessian $= \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial y \partial x}\right]^2 = f_{xx}f_{yy} - [f_{xy}]^2$

Recall $f_{xy} = (f_x)_y = (f_y)_x = f_{yx}$ if 2nd order partials are continuous

Theorem 1: **Two-variable second derivative test**
If 2nd order partials of f are continuous in a neighborhood of a critical point (a, b), then

1. If $det(Hf(a, b)) = \Delta > 0$, then
   if $f_{xx} > 0$, then $f(a, b)$ is a local minimum.
   if $f_{xx} < 0$, then $f(a, b)$ is a local maximum.
2. If $det(Hf(a, b)) = \Delta < 0$, then $f(a, b)$ is neither a local minimum nor a local maximum

Note if $detHf(a, b) = \Delta = 0$, then the 2nd derivative test gives no information.

Example: $f(x, y) = f(x, y) = x^2 + y^2 + pxy$

See chalkboard/https://www.khanacademy.org/math/multivariable-calculus/applic of-multivariable-derivatives/optimizing-multivariable-functions/a/second-partial-derivative-test