Claim: $\nabla F(x,y,z) \;=\; < \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} >$ is perpendicular to surface $F(x,y,z) = 0$

Illustrated by example: Let $F(x,y,z) = f(x,y) - z$

$\nabla F(x,y,z) \;=\; < \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} > \;=\; < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 >$

Then surface defined by $F(x,y,z) = f(x,y) - z = 0$ is $z = f(x,y)$.

Recall $(1, 0, \frac{\partial f}{\partial x})$ and $(0, 1, \frac{\partial f}{\partial y})$ are tangent vectors to surface $z = f(x,y)$.

Thus $\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & \frac{\partial f}{\partial y} \\ 1 & 0 & \frac{\partial f}{\partial x} \end{bmatrix} =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 > \;= \nabla F$ is perpendicular to surface $z = f(x,y)$

Sidenote: If $\mathbf{u} = (u_1, u_2)$, then $(u_1, u_2, D_{\mathbf{u}} f \mathbf{p})$ is tangent to surface $z = f(x,y)$ for any point $\mathbf{p}$ lying on this surface.

That is $(u_1, u_2, D_{\mathbf{u}} f \mathbf{p})$ lies in tangent plane to surface $z = f(x,y)$ at point $\mathbf{p}$.

Proof: $< u_1, u_2, D_{\mathbf{u}} f(\mathbf{p}) > \cdot < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 >= u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} - D_{\mathbf{u}} f(\mathbf{p}) = 0$

since $D_{\mathbf{u}} f \;=\; < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > \cdot < u_1, u_2 > \;=\; \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$

Implicit function theorem.

If $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ are continuous and $\frac{\partial F}{\partial z} \neq 0$ and if $\mathbf{p}$ lies on surface $F(x,y,z) = 0$, then near $\mathbf{p}$, the surface $F(x,y,z) = 0$ coincides with the surface $z = f(x,y)$ for some function f

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Thus $\nabla F(x,y,z) \;=\; < \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} >$ is perpendicular to surface $F(x,y,z) = 0$

That is $\nabla F(x,y,z) \;=\; < \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} >$ is perpendicular to the tangent plane to surface $F(x,y,z) = 0$ at point $\mathbf{p}$.

Thus $< \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} > \cdot (\mathbf{x} - \mathbf{p})$ is the equation of the tangent plant to surface $F(x,y,z) = 0$ at point $\mathbf{p}$.