Calc 1 (one independent variables)

If $y = f(x)$, recall average rate of change = slope of secant line = $\frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x}$

If $y = f(x)$, recall instantaneous rate of change = slope of tangent line:

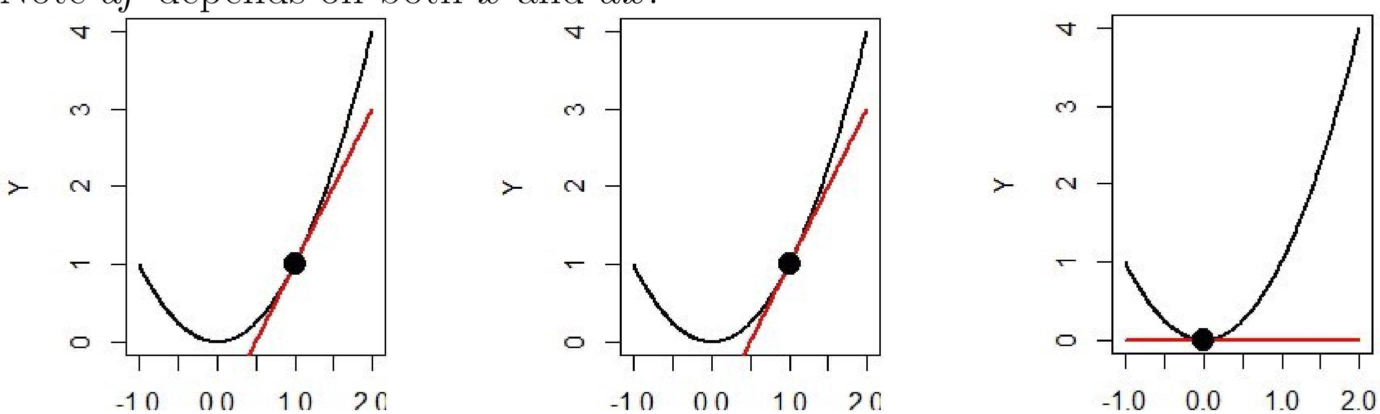
$$\frac{dy}{dx} = \frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

The differential: Useful notation.

$$\frac{df}{dx} = f'(x). \text{ Thus } df = f'(x)dx = f'(x)\Delta x.$$

Note that x is the independent variable. Thus we can let $dx = \Delta x$.

Note df depends on both x and dx.



Change in the dependent variable y (resulting from change in the independent variable x) can be estimated using the differential.

$$\Delta f = \Delta y \sim dy = df = f'(x)dx = f'(x)\Delta x.$$

$$\text{or equivalently, } \frac{\Delta f}{\Delta x} \sim f'(x). \text{ Thus } \Delta f \sim f'(x)\Delta x.$$

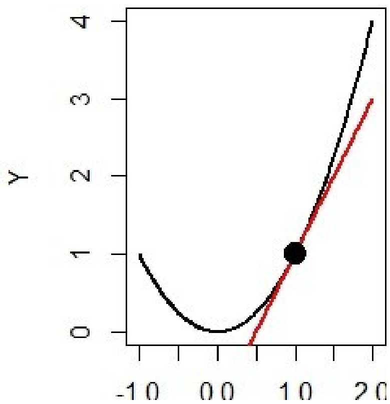
Note: depending on f and $dx = \Delta x$, sometimes this is a good estimate and sometimes this is a bad estimate:

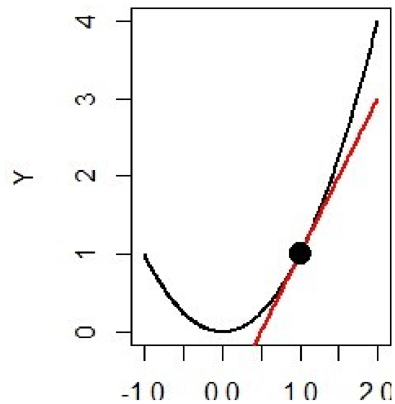
One can also use the differential to estimate $f(x_2)$ if you know $f(x_1)$

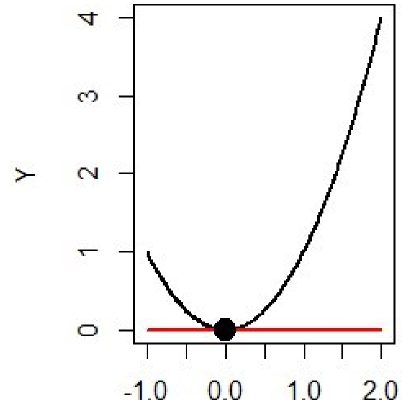
$$f(x_2) = f(x_1) + [f(x_2) - f(x_1)] = f(x_1) + \Delta f \sim f(x_1) + df = f(x_1) + f'(x_1)dx$$

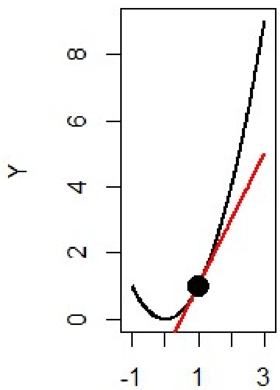
$$\text{Thus } f(x_2) = f(x_1) + f'(x_1)[x_2 - x_1]$$

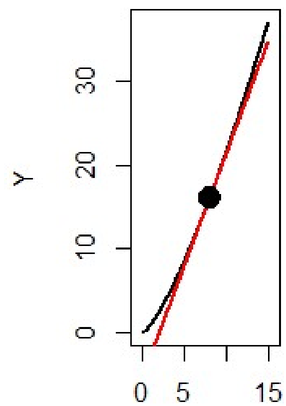
Note equation of tangent line to f at $x = x_1$ is: $y = f(x_1) + f'(x_1)[x - x_1]$











Approximate $10^{\frac{4}{3}}$:

Let $f(x) = x^{\frac{4}{3}}$. Then $f'(x) = \frac{4}{3}x^{\frac{1}{3}}$

Can calculate $f(8)$ and $f'(8)$, so use $x = 8$ to estimate the value of $f(10)$.

Thus $f(10) = f(8) + \Delta y = f(8) + \Delta f \sim f(8) + f'(8)\Delta x$.

Note $dx = \Delta x = 10 - 8 = 2$.

Note $f(8) = 8^{\frac{4}{3}} = 2^4 = 16$. and $f'(8) = \frac{4}{3}(8)^{\frac{1}{3}} = \frac{4}{3}(2) = \frac{8}{3}$

$$10^{\frac{4}{3}} = f(10) = f(8) + [f(10) - f(8)] = f(8) + \Delta f$$

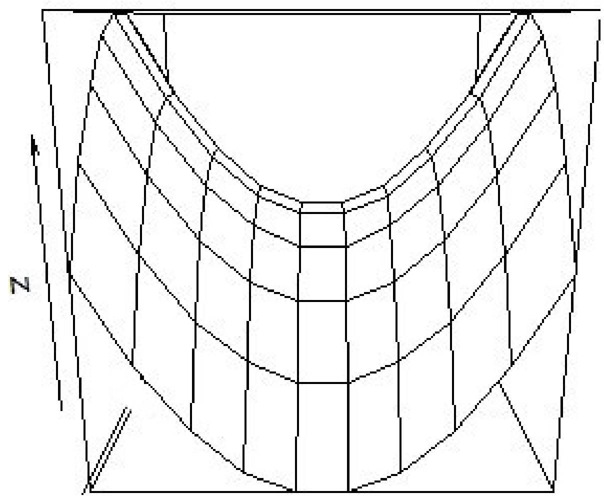
$$\sim f(8) + df = f(8) + f'(8)dx$$

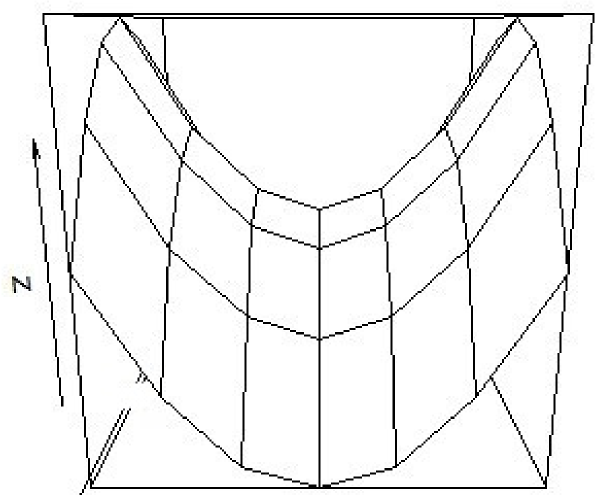
$$= f(8) + f'(8)(10 - 8) = 16 + (\tfrac{8}{3})(2) = \frac{48}{3} + \frac{16}{3} = \frac{64}{3}$$

Note: $\frac{64}{3} = 21.333333...$ while $10^{\frac{4}{3}} = 21.544346900318837217592935665...$

Section 12.6: Math 5 (multiple independent variables)

Suppose $z = f(x, y)$. Then the gradient vector of f is: $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$





Note $\Delta z \sim df = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \cdot (\Delta x, \Delta y) = \nabla f \cdot (\Delta x, \Delta y)$

$$[(x_0 + \Delta x,\ y_0,\ z_0 + \tfrac{\partial f}{\partial x}\Delta x) - (x_0, y_0, z_0)]$$

$$+[(x_0,\ y_0 + \Delta y,\ z_0 + \tfrac{\partial f}{\partial y}\Delta y) - (x_0, y_0, z_0)]$$

$$= [(\Delta x,\ 0,\ \tfrac{\partial f}{\partial x}\Delta x)] + [(0,\ \Delta y,\ \tfrac{\partial f}{\partial y}\Delta y)] = (\Delta x,\ \Delta y,\ \tfrac{\partial f}{\partial x}\Delta x + \tfrac{\partial f}{\partial y}\Delta y)$$

$$= (\Delta x,\ \Delta y,\ \nabla f \cdot (\Delta x, \Delta y)) \sim (\Delta x,\ \Delta y,\ \Delta z)$$

Approximate $(10^{\frac{4}{3}})(24^{\frac{1}{2}})$.

Let $f(x,y) = (x^{\frac{4}{3}})(y^{\frac{1}{2}})$. Use $f(8,25)$ to approximate $f(10,24)$

Note x and y are independent variables, so

$$(dx, dy) = (\Delta x, \Delta y) = (10-8, 24-25) = (2, -1)$$

$f(8,25) = (8^{\frac{4}{3}})(25^{\frac{1}{2}}) = (2^4)(5) = 16(5) = 80$

$\frac{\partial f}{\partial x}(x,y) = \frac{4}{3}x^{\frac{1}{3}}(y^{\frac{1}{2}})$ and $\frac{\partial f}{\partial y}(x,y) = (x^{\frac{4}{3}})\frac{1}{2}y^{-\frac{1}{2}}$

$\frac{\partial f}{\partial x}(8,25) = \frac{4}{3}(8^{\frac{1}{3}})(25^{\frac{1}{2}}) = \frac{4}{3}(2)(5) = \frac{40}{3}$ and

$\frac{\partial f}{\partial y}(8,25) = (8^{\frac{4}{3}})(\frac{1}{2})(25^{-\frac{1}{2}}) = (16)(\frac{1}{2})(\frac{1}{5}) = \frac{8}{5}$ Thus $\nabla f(8,25) = (\frac{40}{3}, \frac{8}{5})$

$(10^{\frac{4}{3}})(24^{\frac{1}{2}}) = f(10,24) = f(8,25) + [f(10,24) - f(8,25)] = f(8,25) + \Delta f$

$\sim f(8,25) + df = f(8,25) + \nabla f \cdot (\Delta x, \Delta y) = 80 + (\frac{40}{3}, \frac{8}{5}) \cdot (2, -1)$

$= 80 + \frac{80}{3} - \frac{8}{5} = \frac{80(15)}{15} + \frac{400}{15} - \frac{24}{15} = \frac{1200}{15} + \frac{376}{15} = \frac{1576}{15}$

Calc 1: Let $z = f(x)$ and $h = \Delta x = dx$, then

$f(a+h) = f(a) + \Delta z \sim f(a) + df = f(a) + f'(a)h$

Thus $f(a+h) - f(a) \sim f'(a)h$. Hence $\frac{f(a+h)-f(a)}{h} \sim f'(a)$

I.e, $\frac{f(a+h)-f(a)-f'(a)h}{h} \sim 0$. I.e, $\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0$

Multivariable Calc: Let $z = f(x,y)$, $\mathbf{a} = (x_0, y_0)$ and $\mathbf{h} = (\Delta x, \Delta y)$.

Then $f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \Delta z \sim f(\mathbf{a}) + df = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h}$

Hence $\lim_{h \to 0} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h}}{|\mathbf{h}|} = 0$

Defn: f is differentiable at $\mathbf{a}$ if these exists a constant vector $\mathbf{c}$ such that

$\lim_{h \to 0} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c} \cdot \mathbf{h}}{|\mathbf{h}|} = 0$ Note $\mathbf{c} = \nabla f(\mathbf{a})$