Defn: A set V together with two operations, called addition and scalar multiplication is a **vector space** if the following vector space axioms are satisfied for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and all scalars, c, d in R.

Vector space axioms:

a.) $\mathbf{u} + \mathbf{v}$ is in V

b.) $c\mathbf{u}$ is in V

c.) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

d.) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

e.) There is a vector, denoted by $\mathbf{0}$, in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V

f.) For each $\mathbf{u}$ in V, there is an element, denoted by $-\mathbf{u}$, in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

g.) $(cd)\mathbf{u} = c(d\mathbf{u})$

h.) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

i.) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

j.) $1\mathbf{u} = \mathbf{u}$

Examples:

1.) R^k with the usual operations of addition and scalar multiplication is a vector space.

2.) The set $M^{k,n}$, the set of all $k \times n$ matrices with the usual operations of addition and scalar multiplication is a vector

Linear Algebra Review: Eigenvalues and Eigenvectors

Defn: λ is an **eigenvalue** of the linear transformation $T : V \to V$ if there exists a <u>nonzero</u> vector $\mathbf{x}$ in V such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

Example: Let $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \mathbf{x}$.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is <u>NOT</u> an eigenvector of A.

MOTIVATION:

Note $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $A \begin{bmatrix} 2 \\ 8 \end{bmatrix} = A(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = A \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$$

Finding eigenvalues:

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ (Note A is a SQUARE matrix).

Then $A\mathbf{x} = \lambda I\mathbf{x}$ where I is the identity matrix.

Thus $\lambda I\mathbf{x} - A\mathbf{x} = (\lambda I - A)\mathbf{x} = \mathbf{0}$

Thus if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Thus $det(\lambda I - A)\mathbf{x} = 0$.

Note that the eigenvectors corresponding to λ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

Thus to find the eigenvalues of A and their corresponding eigenvectors:

Step 1: Find eigenvalues: Solve the equation

$$det(\lambda I - A) = 0 \text{ for } \lambda.$$

Step 2: For each eigenvalue λ_0, find its corresponding eigenvectors by solving the homogeneous system of equations

$$(\lambda_0 I - A)\mathbf{x} = 0 \text{ for } \mathbf{x}.$$

Defn: $det(\lambda I - A) = 0$ is the **characteristic equation** of A.

Thm 3: The eigenvalues of an upper triangular or lower triangular matrix (including diagonal matrices) are identical to its diagonal entries.

Defn: The **eigenspace** corresponding to an eigenvalue λ_0 of a matrix A is the set of all solutions of $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$.

Note: An eigenspace is a vector space

The vector $\mathbf{0}$ is always in the eigenspace.

The vector $\mathbf{0}$ is never an eigenvector.

The number 0 can be an eigenvalue.

Thm: A square matrix is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.