Solving first order differential equation:

Method 1 (sect. 2.2): Separate variables.

Method 2 (sect. 2.1): If linear $[y'(t)+p(t)y(t) = g(t)]$, multiply equation by an integrating factor $u(t) = e^{\int p(t)dt}$.

$$\begin{gathered} y' + py = g \\ y'u + upy = ug \\ (uy)' = ug \\ \int (uy)' = \int ug \\ uy = \int ug \\ \text{etc...} \end{gathered}$$

Method 3 (sect. 2.4): Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n > 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

If $v = \frac{dx}{dt}$, can use the following to simplify (especially if there are 3 variables).

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

integration techniques: u-substitution, integration by parts, partial fractions.

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direction field = slope field = graph of $\frac{dv}{dt}$ in t, v-plane.

*** can use slope field to determine behavior of v including as $t \to \infty$.

Equilibrium Solution = constant solution

stable, unstable, semi-stable.

Solving second order differential equation:

p. 133: $y'' = f(t, y')$, $y'' = f(y, y')$,

Transform to first order: Let $v = y'$.

If needed, note $v' = \frac{dv}{dt} = \frac{dv}{dt}\frac{dy}{dy} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$.

Note this trick sometimes helpful for first order equations.

Ch 3: linear

$ay'' + by' + cy = 0, \quad y = e^{rt}$, then
$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

To solve $ay'' + by' + cy = g_1(t) + g_2(t) + ...g_n(t)$ [**]

1.) Find the general solution to $ay'' + by' + cy = 0$:

$$c_1\phi_1 + c_2\phi_2$$

2.) For each g_i, find a solution to $ay'' + by' + cy = g_i$:

$$\psi_i$$

This includes plugging guessed solution into $ay'' + by' + cy = g_i$ to find constant(s).

The general solution to [**] is

$$c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2 + ...\psi_n$$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

Thm: Suppose that f_1 is a a solution to $ay'' + by' + cy = g_1(t)$ and f_2 is a a solution to $ay'' + by' + cy = g_2(t)$, then $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$

Proof:

Define $L(f) = af'' + bf' + cf$. Note that L is a linear function.

Since f_1 is a solution to $ay''+by'+cy = g_1(t)$, $L(f_1) = af_1'' + bf_1' + cf_1 = g_1(t)$.

Since f_2 is a solution to $ay''+by'+cy = g_2(t)$, $L(f_2) = af_2'' + bf_2' + cf_2 = g_2(t)$.

We will now show that $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

$L(f_1 + f_2) = L(f_1) + L(f_2) = g_1(t) + g_2(t)$. Thus $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

Sidenote: The proofs above work even if a, b, c are functions of t instead of constants.

Existence and Uniqueness

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$\begin{aligned} y' + p(t)y &= g(t), \\ y(t_0) &= y_0 \end{aligned}$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$\begin{aligned} y'' + p(t)y' + q(t)y &= g(t), \\ y(t_0) &= y_0, \\ y'(t_0) &= y'_0 \end{aligned}$$

Definition: The Wronskian of two differential functions, f and g is

$$W(f, g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

sections 3.2, 3.3