A function f is linear if $f(a\mathbf{x}+b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$

Or equivalently f is linear if
1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and 2.) $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

Theorem: If f is linear, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 0.) $f: R \to R$, $g(x) = 2x + 5$ is NOT linear

Proof 1: $g(0) = 5 \neq 0$

Proof 2:
$g(3 \cdot 4) = g(12) = 2(12) + 5 = 29$
$3g(4) = 3[2(4) + 5] = 3[13] = 39$
$29 \neq 39$
Hence $g(3 \cdot 4) \neq 3g(4)$

Proof 3:
$g(0+1) = g(1) = 2(1) + 5 = 7$
$g(0) + g(1) = [2(0) + 5] + [2(1) + 5] = 5 + 7 = 12$
$7 \neq 12$
Hence $g(0+1) \neq g(0) + g(1)$

Example 1.) $f : R \rightarrow R$, $f(x) = 2x$

Proof:
$f(ax+by) = 2(ax+by) = 2ax+2by = af(x)+bf(y)$

Example 3.) D : set of all differential functions $\rightarrow$ set of all functions, $D(f) = f'$

Proof:
$D(af+bg) = (af+bg)' = af'+bg' = aD(f)+bD(g)$

Example 4.) Given a, b real numbers,
I : set of all integrable functions on [a, b] $\rightarrow$ R , $I(f) = \int_a^b f$

Proof:
$I(sf+tg) = \int_a^b sf+tg = s\int_a^b f+t\int_a^b g = sI(f)+tI(g)$

Example 5.) The inverse of a linear function is linear (when the inverse exists).

Proof: Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$.

Then $f(c) = x$ and $f(d) = y$ and
$f(ac+bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax+by) = ac+bd = af^{-1}(x) + bf^{-1}(y)$.

Example 6.) D : set of all twice differential functions $\to$ set of all functions, $L(f) = af'' + bf' + cf$

Proof:

$$
\begin{aligned}
L(sf + tg) &= a(sf + tg)'' + b(sf + tg)' + c(sf + tg) \\
&= saf'' + tag'' + sbf' + tbg' + scf + tcg \\
&= s(af'' + bf' + cf) + t(ag'' + bg' + cg) \\
&= sL(f) + tL(g)
\end{aligned}
$$

Example 7.) The LaPlace transform $\mathcal{L}$ is linear

$\mathcal{L}$: set of all functions satisfying hypothesis of thm 6.1.2 $\to$ set of all functions,

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = F(s)$$

Thm 6.1.2: The Laplace transform $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = F(s)$ exists for all $s > a$ if

1.) f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.

2.) There exist constants K, a, M such that $|f(t)| \leq Ke^{at}$ for all $t \geq M$.

Theorem: Suppose f, f', ..., $f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on $0 \le t \le A$. Suppose there exists constance K, a, and M such that $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}$, ..., $|f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}(f^{(n)})$ exists for $s > a$ and is given by

$$\mathcal{L}(f^{(n)})$$
$$= s^n \mathcal{L}(f) - s^{n-1} f(0) - ... - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

LaPlace Transform

The LaPlace Transform is a method to change a differential equation to a linear equation.

Example:
Solve $y'' + 3y' + 4y = 0$, $y(0) = 5$, $y'(0) = 6$.

1.) Take the LaPlace Transform of both sides of the equation:

2.) Use the fact that the LaPlace Transform is linear:

3.) Use thm to change this equation into an algebraic equation:

$$\mathcal{L}(f^{(n)})$$
$$= s^n\mathcal{L}(f) - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

3.5) Substitute in the initial values.

4.) Solve the algebraic equation for $\mathcal{L}(y)$

Some algebra implies $\mathcal{L}(y) =$

5.) Solve for y by taking the inverse LaPlace transform of both sides (use a table):

To find the inverse LaPlace transform, you may need to use that the inverse LaPlace transform in linear. You may also need to use partial fractions or other methods in order to write the right-hand side of (*) as a sum of functions whose inverse LaPlace transforms are known.

Calculus pre-requisites you must know.

Derivative = slope of tangent line = rate.

Integral = area between curve and x-axis (where area can be negative).

A function is (Riemann) integrable if this area can be calculated using rectangles as in first year calculus.