No external force: homogeneous equation

$mu''(t) + \gamma u'(t) + ku(t) = 0, \quad m, \gamma, k \geq 0$

$mr^2 + \gamma r + k = 0$ implies $r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$

Case 1: $\gamma^2 - 4km < 0$

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{-\gamma \pm \sqrt{(-1)(4km - \gamma^2)}}{2m} = \frac{-\gamma \pm i\sqrt{4km - \gamma^2}}{2m}$$

$$u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t) = e^{-\frac{\gamma t}{2m}}Rcos(\mu t - \delta)$$

where $\mu = \frac{\sqrt{4km - \gamma^2}}{2m}$

and $A = Rcos(\delta)$, $B = Rsin(\delta)$ ($R = \sqrt{A^2 + B^2}$, $tan\delta = \frac{B}{A}$)

Note A, B and hence R, δ depend on initial conditions.

μ does not depend on initial conditions
i.e., $\mu =$ quasi frequency and $\frac{2\pi}{\mu} =$ quasi period do not depend on initial conditions.

$\gamma = 0$: No damping

$$u(t) = Acos\mu t + Bsin\mu t = Rcos(\mu t - \delta)$$

$$\mu = \frac{\sqrt{4km}}{2m} = \sqrt{\frac{k}{m}} = \omega_0$$

Example: $y'' + 9y = 0$

With damping: $\gamma \neq 0$,

Small damping: $\gamma < 2\sqrt{km}$ (i.e., $\gamma^2 - 4km < 0$).

$y'' + y' + 9y = 0$

$y'' + 3y' + 9y = 0$

$u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t) = e^{-\frac{\gamma t}{2m}}Rcos(\mu t - \delta)$

where $\mu = \frac{\sqrt{4km-\gamma^2}}{2m}$

and $A = Rcos(\delta)$, $B = Rsin(\delta)$ ($R = \sqrt{A^2 + B^2}$, $tan\delta = \frac{B}{A}$)

Note A, B and hence R, δ depend on initial conditions. μ does not depend on initial conditions.

Case 2: Critical Damping: $\gamma = 2\sqrt{km}$

$\gamma^2 - 4km = 0$: $u(t) = (A + Bt)e^{r_1 t}$

$y'' + 6y' + 9y = 0$

Case 3: Over-damped: $\gamma > 2\sqrt{km}$

$\gamma^2 - 4km > 0$: $u(t) = Ae^{r_1 t} + Be^{r_2 t}$

$y'' + 7y' + 9y = 0$

$y'' + 10y' + 9y = 0$