

**22M:034, FALL 2003**

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Computer assignment 3, due December 12

At the beginning of any MAPLE session related to this assignment, type

```
with(plots);  
with(DEtools);
```

Remember that when you reopen a session you have worked on before, you have to step through the above instructions (or “execute worksheet”) again to put MAPLE in the right logical state.

In this assignment, we’ll try to develop, in an experimental way, the beginnings of a geometric approach to equations whose solutions cannot be written down in closed form. These equations resist “formula” solution in part because of their *nonlinearity*. However, we shall work with nonlinear variants of some of the linear equations we have been studying in class (specifically, *oscillator* equations) – and some of the properties of solutions of these linear oscillator equations survive in the nonlinear setting.

Try the following instruction:

```
(A)      fieldplot([y,-x-x^3],x=-1..1,y=-1..1,arrows=SLIM,  
                axes=BOXED,title="hard oscillator field plot");
```

You’ll get a plot with little arrows swirling around the origin and tracing out oval shapes. What happened is this: the instruction is meant to give an idea of the trajectories of the *hard oscillator* equation

$$(1) \quad x'' + x + x^3 = 0.$$

Here  $x = x(t)$  is a function of the time  $t$ , and  $' = d/dt$ . Note the *nonlinear* term  $x^3$  – if this were not there, we would just have the oscillator equation  $x'' + x = 0$ , with closed form general solution  $c_1 \cos(t) + c_2 \sin(t)$ .

A fruitful way to think of (1) is as a *system* of equations. If we give  $x'$  the name  $y$ , the system is

$$(2) \quad \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x - x^3 \end{pmatrix}.$$

The top entry in this column vector equation is just the definition of  $y$ , and the bottom entry is the original differential equation (1); in this way we rewrite (1) in a first order vector equation. One of the simplest things we can do with this is to draw pictures, in an  $xy$ -plane, of the vectors  $\begin{pmatrix} y \\ -x - x^3 \end{pmatrix}$  for different values of  $\begin{pmatrix} x \\ y \end{pmatrix}$  to get a *vector field*; this is what the `fieldplot` instruction did. The resulting vector field tells us which way the trajectories are headed, and allows us to draw them. We have not *solved* the equation (1) (which we can't do in closed form anyway). But what we have done is to get a picture of how solutions move in this  $xy$ -plane, called the *phase plane*. Note that the time  $t$  is not pictured; one imagines a particle moving around in the phase plane, and following the arrows. Thus a moving picture would really be required in order to capture the full story of what is going on with  $t$ ,  $x = x(t)$ , and  $y = y(t)$ .

The phase plane, and higher (even infinite!) dimensional versions of it, are important theoretical tools in Physics, and important practical tools in applications of Physics and Mathematics.

We can demystify the phase plane slightly by recalling our old *reduction of order* trick. Since  $t$  is missing from the equation (1), we proceed by giving a name to  $x'$  (we've already assigned it the name  $y$  above). We then note that

$$x'' = \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = y \frac{dy}{dx}.$$

Equation (1) becomes the first-order equation

$$(3) \quad y \frac{dy}{dx} = -x - x^3.$$

Our “fieldplot” in the phase plane is nothing but the direction field for the first-order equation (3).

We can even solve (3) by separating variables:

$$(4) \quad \begin{aligned} y \, dy &= (-x - x^3) dx, \\ \frac{1}{2} y^2 &= -\frac{1}{2} x^2 - \frac{1}{4} x^4 + C, \\ \frac{1}{2} y^2 + \frac{1}{2} x^2 + \frac{1}{4} x^4 &= C. \end{aligned}$$

The last line above is *conservation of energy* for the (hard) nonlinear oscillator. Of course this does not give us a formula for  $x$  as a function of  $t$ , but it does give much valuable information.

The meaning of the various parts of the MAPLE instruction we gave are as follows. `[y, -x - x^3]` represents the right hand side  $\begin{pmatrix} y \\ -x - x^3 \end{pmatrix}$  of the differential equation. `x=-1..1, y=-1..1` gives the ranges in which we wish MAPLE to plot. The other instructions are devoted to the appearance of the plot, and are more or less self-explanatory. Be careful to use the correct quotation marks in the `title` instruction. Instead of the double quote `"` you can also use the single quote `'` from the upper left of the keyboard – but not the one from the lower right. Note that the arrows have different lengths – the length is proportional to the length of the right hand side vector  $\begin{pmatrix} y \\ -x - x^3 \end{pmatrix}$ . Thus we get an idea of how fast the trajectory is traced out, at least relative to other trajectories.

A little more sophisticated visually is

```
(B)      dfieldplot([diff(x(t),t)=y(t),diff(y(t),t)=-x(t)-x(t)^3],
            [x(t),y(t)],0.1..0.2,x=0..1,y=0..1, arrows=SLIM,axes=BOXED,
            title='hard oscillator direction field');
```

Here we lose the ability to judge the speed of the trajectories by looking at the length of the arrows. There is a “ $t$ -range” argument `0.1..0.2` in the instruction, but this would really only make a difference if  $t$  appeared explicitly on the right hand side of the differential equation. There are several visual options in the menus above the plot however, and one can see the coordinates of the point under the mouse cursor. Note that we need to specify the plot axes with the argument `[x(t),y(t)]`.

Here's a way we can make use of the energy function

$$(5) \quad E = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4$$

for the hard oscillator. Recall from (4) that the energy of a solution is *conserved* – it stays constant as the time moves.

```
(C)      DEplot([diff(x(t),t)=y(t),diff(y(t),t)=-x(t)-x(t)^3],
            [x(t),y(t)],-2..2,{[0,0,1],[0,1,0]},color=y^2+x^2+(1/2)*x^4,
            arrows=THICK,title='hard oscillator, color=energy');
```

We get a plot where energy levels are *isochromes* (curves with a constant color). Note that the quantity  $2E = y^2 + x^2 + \frac{1}{2}x^4$  appeared in our instruction instead of  $E$ . This is purely

a convenience – we decided to use  $2E$  instead of  $E$  because it was easier to type, and  $2E$  does just as good a job as  $E$  (or any positive multiple of  $E$ ) in telling us where the higher and lower energy levels are.

The argument  $\{[0, 0, 1], [0, 1, 0]\}$  asks MAPLE to draw in the trajectories passing through  $(x, y) = (0, 1)$  and  $(1, 0)$ ; i.e.  $(t, x, y) = (0, 0, 1)$  and  $(0, 1, 0)$ . (What we set  $t$  equal to at these points will be irrelevant – why?) If we have an equation with a conserved energy, there's an easy way to get the *contour curves*, or *level sets* of the energy, on which trajectories must lie:

(D) 

```
contourplot(x^2+y^2+(1/2)*x^4,x=0..2,y=0..2,
title='contour plot, hard oscillator energy');
```

With a `phaseportrait` instruction, MAPLE actually does a numerical solution of the differential equation, for given initial conditions, and plots in the phase plane. For the hard oscillator, our instruction might look like this:

(E) 

```
phaseportrait([diff(x(t),t)=y(t),diff(y(t),t)=-x(t)-x(t)^3],
[x(t),y(t)],0..8,[0,0,1],[0,1,0]),scene=[x,y],
stepsize=.1,title='hard oscillator phase portrait').
```

There is an interesting tool hidden here: by making all the trajectories run for a certain amount of time (the  $0..8$  in (E) for example), we can get an idea of which trajectories are being traced out faster – the longer the trajectory segment in the phase portrait, the faster one moves along that trajectory.

### Assignment:

Taking your 8-digit University ID number, let

$a =$  the 6<sup>th</sup> digit of your ID number,

$b =$  the 7<sup>th</sup> digit of your ID number,

$c = \frac{1}{5}$ (the 8<sup>th</sup> digit of your ID number).

In all cases, if the digit in question is 0, use the value 10 instead.

Using instruction (C) and the sample MAPLE sessions posted on the course Web page as

prototypes, do a DEplot for each of the following:

the hard oscillator	$x'' + ax + bx^3 = 0,$
the inhomogeneous hard oscillator	$x'' + ax + bx^3 = b \cos\left(\frac{a}{10}t\right),$
the soft oscillator	$x'' + ax - bx^3 = 0,$
the pendulum	$x'' + a \sin(bx) = 0$
the damped hard oscillator	$x'' + cx' + ax + bx^3 = 0,$
the damped soft oscillator	$x'' + cx' + ax - bx^3 = 0,$
the damped pendulum	$x'' + cx' + a \sin(bx) = 0,$
an equation from Economics	$x'' - x' + x + \frac{a}{10}(x')^3 = 0.$

Use the `title=` argument to put descriptive titles on your plots. Try to adjust the parameters in the instructions to get the most revealing and instructive pictures – the parameters that will work best depend on your individual  $a, b, c$  of course. Make sure you get trajectories of each basic type (more on this in class). This is the art of it – tweaking parameters to get the best picture, combining educated guesses with trial and error. By adjusting the length of time that solutions run, try to get the picture to give information about the relative speed at which trajectories are being traversed. For the equations with damping, see whether you can get good pictures of spiraling behavior.

Note that you need to figure out what the correct *energy* is for most of these equations. It's possible to do this by following the arguments in the first part of this handout. A hint: In going from an undamped to a damped equation, the energy *function* doesn't change. However, for a damped equation, the energy *decreases* rather than sitting still (as it does for an undamped equation). This phenomenon, *dissipation*, manifests itself in the phase plane picture – there will be trajectories that exhibit a “spiraling in” behavior. There will be more information in class on energy functions.