Math 34 Differential Equations Final Exam
December 15, 2003 SHOW ALL WORK

Choose 10 of the following 11 problems. If you choose fewer than 10, the points for the remaining problems will be appropriately averaged. 3 of the problems have 1 point extra credit available (indicated by [1-EC]). You can do the extra credit even if you do not choose these problems.

[3] 1a.) Without using the LaPlace transform, solve the following initial value problem:

$$y'' + 2y' + y = 0, \ y(0) = 0, \ y'(0) = 4$$

Suppose $y = e^{rt}$. Then $y' = re^{rt}, \ y'' = r^2e^{rt}$

$r^2e^{rt} + 2re^{rt} + e^{rt} = 0$ Hence $r^2 + 2r + 1 = 0$. Thus, $(r+1)^2 = 0$. Hence $r = -1$

Hence general solution is $y(t) = c_1e^{-t} + c_2te^{-t}$

$y(0) = 0 : 0 = c_1 + 0$. Thus $c_1 = 0$

$y'(0) = 4 : y' = c_2(e^{-t} - te^{-t})$. Thus $4 = c_2(1 - 0) = c_2$

Thus, $y(t) = 4te^{-t}$ is the solution to the initial value problem.

Answer 1a.) $\underline{4te^{-t}}$

[4] 1b.) Without using the LaPlace transform, find the general solution to the differential equation $y'' + 2y' + y = e^{-t}$

Note c_1e^{-t} and c_2te^{-t} are both solutions to the homogenous equation (without the initial values). Hence they cannot be solutions to the non-homogeneous equation.

Thus guess $y = At^2e^{-t}$. Then $y' = A(2te^{-t} - t^2e^{-t})$
$y'' = A(2e^{-t} - 2te^{-t} - 2te^{-t} + t^2e^{-t}) = A(2e^{-t} - 4te^{-t} + t^2e^{-t})$

From $y'' + 2y' + y = e^{-t}$, we get

$A(2e^{-t} - 4te^{-t} + t^2e^{-t}) + 2A(2te^{-t} - t^2e^{-t}) + At^2e^{-t} = e^{-t}$

$A(2e^{-t} - 4te^{-t} + t^2e^{-t} + 4te^{-t} - 2t^2e^{-t} + t^2e^{-t}) = e^{-t}$

$A(2e^{-t}) = e^{-t}$. Hence $2A = 1$ and $A = \frac{1}{2}$

Thus $\frac{1}{2}t^2e^{-t}$ is a solution to $y'' + 2y' + y = e^{-t}$

Answer 1b.) $\underline{\frac{1}{2}t^2e^{-t} + c_1e^{-t} + c_2te^{-t}}$

Note if you guessed the wrong solution, but did everything else correctly, you could still earn 3 - 4pts for this problem depending on whether and how well you explained how your answer could not have been correct and how to get the right answer.

[3] 1c.) Without using the LaPlace transform, solve the following initial value problem:

$$y'' + 2y' + y = e^{-t}, \; y(0) = 0, \; y'(0) = 4$$

$y = \frac{1}{2}t^2e^{-t} + c_1e^{-t} + c_2te^{-t}$

$y(0) = 0: \quad 0 = 0 + c_1 + 0$. Hence $c_1 = 0$.

$y' = [te^{-t} - \frac{1}{2}t^2e^{-t}] + c_2[e^{-t} - te^{-t}]$

$y'(0) = 4: \quad 4 = [0 - 0] + c_2[1 - 0]$. Hence $c_2 = 4$.

Answer 1c.) $\underline{y(t) = \frac{1}{2}t^2e^{-t} + 4te^{-t}}$

Note the above problem graded based upon your answer to 1b. Hence you could get full credit for this problem even if your answer to 1b was wrong.

Note also that although these are the same solutions for c_1 and c_2 as in 1a, this is not usually the case, so you can't just take those solutions for the coefficients in 1c.

[10] 2.) Use the LaPlace transform to solve the following initial value problem:

$$y'' + 2y' + y = e^{-t}, \; y(0) = 0, \; y'(0) = 4$$

$\mathcal{L}(y'') + 2\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(e^{-t})$

$s^2\mathcal{L}(y) - sy(0) - y'(0) + 2[s\mathcal{L}(y) - y(0)] + \mathcal{L}(y) = \frac{1}{s+1}$

$(s^2 + 2s + 1)\mathcal{L}(y) - 4 = \frac{1}{s+1}$

$(s^2 + 2s + 1)\mathcal{L}(y) = 4 + \frac{1}{s+1}$

$(s + 1)^2\mathcal{L}(y) = 4 + \frac{1}{s+1}$

$\mathcal{L}(y) = \frac{4}{(s+1)^2} + \frac{1}{(s+1)^3}$

$y = 4\mathcal{L}^{-1}(\frac{1}{(s+1)^2}) + \frac{1}{2}\mathcal{L}^{-1}(\frac{2}{(s+1)^3})$

$y = 4te^{-t} + \frac{1}{2}t^2e^{-t}$

Answer 2.) $\underline{y(t) = 4te^{-t} + \frac{1}{2}t^2e^{-t}}$

[1-EC] 2a.) How can you use the LaPlace transform to find the general solution to a differential equation.

Use variables for $y(0)$ and $y'(0)$. For example, let $y(0) = y_0$ and $y'(0) = y_0'$. Since these can be treated as constants, you can then solve the differential equation as you normally would using the LaPlace transform.

[10] 3.) Find the inverse LaPlace transform of $e^{-3s}\frac{s}{s^2+8s+18}$

$\mathcal{L}^{-1}(e^{-3s}\frac{s}{s^2+8s+18}) = \mathcal{L}^{-1}(e^{-3s}\mathcal{L}(f(t))) = u_3(t)f(t-3)$ where

$\mathcal{L}(f(t)) = \frac{s}{s^2+8s+18} = \frac{s}{(s+4)^2+2} = \frac{s+4-4}{(s+4)^2+2} = \frac{s+4}{(s+4)^2+2} - \frac{4}{\sqrt{2}}\frac{\sqrt{2}}{(s+4)^2+2}$

Hence $f(t) = e^{-4t}cos(\sqrt{2}t) - \frac{4}{\sqrt{2}}e^{-4t}sin(\sqrt{2}t)$

Answer 3.) $\underline{u_3[e^{-4(t-3)}cos(\sqrt{2}(t-3)) - \frac{4}{\sqrt{2}}e^{-4(t-3)}sin(\sqrt{2}(t-3))]}$

[10] 4.) Use the definition and not the table to find the LaPlace transform of $f(t) = 3t^2$. CLEARLY indicate when you are taking a limit.

$\int_0^\infty e^{-st}(3t^2)dt = 3t^2\frac{e^{-st}}{-s}|_0^\infty - \int_0^\infty 6t\frac{e^{-st}}{-s}$

$= [lim_{t\to\infty}3t^2\frac{e^{-st}}{-s} - 3(0)^2\frac{e^0}{-s}] - [6t\frac{e^{-st}}{s^2}|_0^\infty - \int_0^\infty 6\frac{e^{-st}}{s^2}]$

$= [0-0] - [(lim_{t\to\infty}6t\frac{e^{-st}}{s^2} - 6(0)\frac{e^0}{s^2}) - 6\frac{e^{-st}}{-s^3}|_0^\infty]$

$= -[(0-0) - (lim_{t\to\infty}6\frac{e^{-st}}{-s^3} - 6\frac{e^0}{-s^3})]$

$= -6\frac{1}{-s^3} = \frac{6}{s^3}$

Let $u = 3t^2,\ dv = e^{-st}$

$du = 6t,\ v = \frac{e^{-st}}{-s}$

$d^2u = 6,\ \int v = \frac{e^{-st}}{s^2}$

Answer 4.) $\underline{\frac{6}{s^3}}$

[5] 5a.) Find the inverse LaPlace transform $\frac{1}{(s-3)(s^2-6s+10)}$ by using the convolution integral. Leave your answer in terms of a convolution integral:

$$\mathcal{L}^{-1}(\frac{1}{(s-3)(s^2-6s+10)}) = \mathcal{L}^{-1}(\frac{1}{(s-3)} \cdot \frac{1}{(s-3)^2+1}) = e^{3t} * e^{3t}sin(t) = \int_0^t e^{3(t-s)}e^{3s}sin(s)ds$$

OR

$$\mathcal{L}^{-1}(\frac{1}{(s-3)(-5s+10)}) = -\frac{1}{5}\mathcal{L}^{-1}(\frac{1}{(s-3)} \cdot \frac{1}{(s-2)}) = -\frac{1}{5}e^{3t} * e^{2t} = -\frac{1}{5}\int_0^t e^{3(t-s)}e^{2s}ds$$

Answer 5a.) $\underline{\int_0^t e^{3(t-s)}e^{3s}sin(s)ds \text{ OR } -\frac{1}{5}\int_0^t e^{3(t-s)}e^{2s}ds}$

[5] 5b.) Evaluate the convolution integral obtained in 3a.

$$\int_0^t e^{3(t-s)}e^{3s}sin(s)ds = \int_0^t e^{3t}e^{-3s}e^{3s}sin(s)ds = e^{3t}\int_0^t sin(s)ds = e^{3t}(-cos(s))|_0^t$$

$$= e^{3t}(-cos(t) - (-cos(0))) = e^{3t}(-cos(t) + 1)$$

OR

$$-\frac{1}{5}\int_0^t e^{3(t-s)}e^{2s}ds = -\frac{1}{5}\int_0^t e^{3t}e^{-3s}e^{2s}ds = -\frac{1}{5}e^{3t}\int_0^t e^{-s}ds = -\frac{1}{5}e^{3t}(-e^{-s})|_0^t$$

$$= -\frac{1}{5}e^{3t}(-e^{-t} - (-e^0)) = -\frac{1}{5}e^{3t}(-e^{-t} + 1)$$

Answer 5b.) $\underline{e^{3t}(-cos(t) + 1) \text{ OR } -\frac{1}{5}e^{3t}(-e^{-t} + 1)}$

[10] 6.) Solve the following initial value problem: $\frac{y'}{t^2} = -3y + 1,\ y(0) = 0$

Method 1: separation of variables

$\frac{dy}{dt} = t^2(-3y + 1)$

$\int \frac{dy}{-3y+1} = \int t^2 dt$

$\frac{1}{-3} ln|-3y + 1| = \frac{1}{3}t^3 + C$

$ln|-3y + 1| = -t^3 + C$

$|-3y + 1| = e^{-t^3}e^C$

$-3y = Ce^{-t^3} - 1$

$y = Ce^{-t^3} + \frac{1}{3}$

Method 2: Integrating Factor

$y' + 3t^2y = t^2.$ Let $u = e^{\int 3t^2 dt} = e^{t^3}$

$e^{t^3}y' + 3t^2e^{t^3}y = t^2e^{t^3}$

$(e^{t^3}y)' = t^2e^{t^3}$

$\int (e^{t^3}y)' = \int t^2e^{t^3}dt$. Integration by substitution: Let $u = t^3$, $du = 3t^2dt$

$e^{t^3}y = \frac{1}{3}\int e^u du$

$e^{t^3}y = \frac{1}{3}e^u + C = \frac{1}{3}e^{t^3} + C$

$y = \frac{1}{3} + Ce^{-t^3}$

$y(0) = 0$: $0 = \frac{1}{3} + C$. Hence $C = -\frac{1}{3}$

Answer 6.) $\underline{y = -\frac{1}{3}e^{-t^3} + \frac{1}{3}}$

[10] 7.) Draw the direction field for the differential equation $y' = (y - 2)(y + 2)$. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

If $y(0) > 2$, then $lim_{t\to\infty}y(t) = +\infty$

If $y(0) = 2$ or -2, then $lim_{t\to\infty}y(t) = 2$

If $y(0) < 2$, then $lim_{t\to\infty}y(t) = -2$

[5] 8a.) Transform $x'' - 2x' - 3x = e^t$ into a system of first order differential equations.

$x_1 = x$

$x_1' = x_2 = x'$

$x_2' = x''$

Answer 8a.) $\underline{x_1' = x_2, x_2' - 2x_2 - 3x_1 = e^t}$

[5] 8b.) Use Euler's formula to write e^{4-2i} in the form of $a + ib$.

$e^{4-2i} = e^4 e^{-2i} = e^4(cos(-2) + isin(-2)) = e^4 cos(-2) + ie^4 sin(-2) = e^4 cos(2) - ie^4 sin(2)$

Answer 8b.) $\underline{e^4 cos(-2) + ie^4 sin(-2) \text{ or } e^4 cos(2) - ie^4 sin(2)}$

9.) Circle T for true and F for false.

[2] 9a.) Given an initial value, there always exists a unique solution to any second order differential equation.

F

[2] 9b.) Given an initial value, there always exists at least one solution to any second order differential equation.

F

[2] 9c.) If there is no damping and no external force, then the general solution to a mechanical vibration problem with constant spring force must be of the form $c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t)$

T

[1] 9d.) $y = c_1 e^{3t} + c_2 t e^{3t} + 4cost$ is a possible general solution to a mechanical vibration problem with damping.

F

[4 or 1-EC] 9e.) Explain your answer to problem 9d.

Short answer $3 > 0$.

Long answer: The long term behaviour should not be affected by the initial conditions. Since $3 > 0$, the long term behaviour of $y = c_1 e^{3t} + c_2 t e^{3t} + 4cost$ is definitely affected by the values of c_1 and c_2 and hence by initial conditions. Recall that when damping is present, the possible solutions for the homogeneous part of the answers (i.e., the part affected by initial conditions) must be of one of the following forms: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$, or $e^d(cos(\omega_0 t) + sin(\omega_0 t))$ where $r_1, r_2, d < 0$.

[10] 10a.) A 2 kg mass stretches a spring .1m. If the mass is pulled down an additional .2m and released, and if there is no damping, determine the position of the mass at any time t

$mg - kL = 0$ implies $k = \frac{mg}{L} = \frac{2(9.8)}{.1} = 196$

$2u'' + 196u = 0$, $u(0) = .2$, $u'(0) = 0$.

$u'' + 98u = 0$

$r^2 + 98 = 0$

$r = i\sqrt{98}$

$u(t) = c_1 cos(\sqrt{98}t) + c_2 sin(\sqrt{98}t)$

$u(0) = .2$: $0.2 = c_1$

$u'(t) = -c_1\sqrt{98}sin(\sqrt{98}t) + c_2\sqrt{98}cos(\sqrt{98}t)$

$u'(0) = 0$: $0 = \sqrt{98}c_2$, $c_2 = 0$.

Answer 10a.) $\underline{u(t) = 0.2cos(\sqrt{98}t)}$

[1-EC] 10b.) Do the initial conditions affect the long-term behavior of the motion of the mass?

Yes (since there is no damping).

[10] 11.) Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously. Find the time T required for the original sum to double in value as a function of r. Assume that the rate of change of the value of the investment is equal to the interest rate r times the current value of the investment $S(t)$.

$\frac{dS}{dt} = rS(t)$ $\qquad \frac{dS}{S(t)} = rdt$ $\qquad ln|S| = rt + C$ $\qquad |S| = e^{rt}e^C$

$S = Ce^{rt}$

$S(0) = S_0$: $S_0 = Ce^0 = C$

$S = S_0e^{rt}$

$2S_0 = S_0e^{rt}$, $\qquad 2 = e^{rt}$, $\qquad ln(2) = rt$

$t = \frac{ln(2)}{r}$

Answer 11.) $\underline{t = \frac{ln(2)}{r}}$