6.5: Impulse functions

Unit impulse function = Dirac delta function is a generalized function with the properties

$$\delta(t) = 0, \quad t \neq 0 \qquad\qquad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\mathcal{L}(\delta(t - t_0)) = e^{-st_0}$$

Let $d_k(t) = \begin{cases} \frac{1}{2k} & -k < t < k \\ 0 & t \leq -k \text{ or } t \geq k \end{cases}$

Note $lim_{k\to 0} d_k(t) = 0$ if $t \neq 0$

and $lim_{k\to 0} \int_{-\infty}^{\infty} d_k(t) = lim_{k\to 0} 1 = 1 = \int_{-\infty}^{\infty} \delta(t)dt$

$$\mathcal{L}(\delta(t - t_0)) = lim_{k\to 0}\mathcal{L}(d_k(t - t_0))$$

$$= lim_{k\to 0} \int_0^{\infty} e^{-st} d_k(t - t_0)dt$$

$$= lim_{k\to 0} \frac{1}{2k} \int_{t_0-k}^{t_0+k} e^{-st}dt$$

$$= lim_{k\to 0} \frac{-1}{2sk} e^{-st}\Big|_{t_0-k}^{t_0+k}$$

$$= lim_{k\to 0} \frac{1}{2sk} e^{-st_0}\left(e^{sk} - e^{-sk}\right)$$

$$= lim_{k\to 0} \frac{sinh(sk)}{sk} e^{-st_0}$$

$$= lim_{k\to 0} \frac{scosh(sk)}{s} e^{-st_0} \;=\; e^{-st_0}$$

$sin(t) = \frac{e^{it} - e^{-it}}{2i}$ $\qquad$ $cos(t) = \frac{e^{it} + e^{-it}}{2}$

$sinh(t) = \frac{e^{t} - e^{-t}}{2}$ $\qquad$ $cosh(t) = \frac{e^{t} + e^{-t}}{2}$

$[sinh(t)]' =$ $\qquad$ $[cosh(t)]' =$

$sinh(0) = \frac{e^{0} - e^{0}}{2} = 0$ $\qquad$ $cosh(0) = \frac{e^{0} + e^{0}}{2} = 1$

Intro to Group Theory

Define the $\cdot$ product on R^2 by

$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 - y_1 x_2)$

Note $\cdot$ is

1.) commutative:
$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 - y_1 x_2)$
$= (x_2 x_1 - y_2 y_1, x_2 y_1 - y_2 x_1) = (x_2, y_2) \cdot (x_1, y_1)$

2.) associative: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$

3.) distributive w.r.t +: $f \cdot (g_1 + g_2) = f \cdot g_1 + f \cdot g_2$

4.) $(x_1, y_1) \cdot (0, 0) = (0, 0)$

Note $(0, 1) \cdot (0, 1) = (-1, 0)$

6.6: The Convolution Integral

Defn: The convolution of f and g is the function $f * g$ defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t f(x)g(t-x)dx$$

Note $*$ is

1.) commutative: $f * g = g * f$

2.) associative: $(f * g) * h = f * (g * h)$

3.) distributive w.r.t $+$: $f * (g_1 + g_2) = f * g_1 + f * g_2$

4.) $f * 0 = 0 * f = 0$

Example: $cos(t) * 1 =$

Example: $t * t \not\geq 0$

Thm: $\mathcal{L}((f*g)(t)) = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t))$

Proof:

$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \int_0^\infty e^{-sy} f(y)dy \int_0^\infty e^{-sx} g(x)dx$

$= \int_0^\infty [\int_0^\infty e^{-sy} f(y)dy] e^{-sx} g(x)dx$

$= \int_0^\infty [\int_0^\infty e^{-sy} f(y) e^{-sx} g(x)dy]dx$

$= \int_0^\infty [\int_0^\infty e^{-s(y+x)} f(y)g(x)dy]dx$

$= \int_0^\infty [\int_0^\infty e^{-s(y+x)} f(y)g(x)dx]dy$

Let $t = x + y$, $dt = dx$

$= \int_0^\infty [\int_y^\infty e^{-st} f(y)g(t-y)dt]dy$

$= \int_0^\infty [\int_0^t e^{-st} f(y)g(t-y)dy]dt$

$= \int_0^\infty e^{-st} [\int_0^t f(y)g(t-y)dy]dt$

$= \int_0^\infty e^{-st} (f*g)(t)dt$

$= \mathcal{L}(f*g)$

Example: $\mathcal{L}^{-1}(\frac{1}{s(s-a)}) =$