

## 6.5: Impulse functions

Unit impulse function = Dirac delta function is a generalized function with the properties

$$\delta(t) = 0, \quad t \neq 0 \qquad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\mathcal{L}(\delta(t - t_0)) = e^{-st_0}$$

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$$\text{Let } d_k(t) = \begin{cases} \frac{1}{2k} & -k < t < k \\ 0 & t \leq -k \text{ or } t \geq k \end{cases}$$

Note  $\lim_{k \rightarrow 0} d_k(t) = 0$  if  $t \neq 0$

$$\text{and } \lim_{k \rightarrow 0} \int_{-\infty}^{\infty} d_k(t) dt = \lim_{k \rightarrow 0} 1 = 1 = \int_{-\infty}^{\infty} \delta(t) dt$$

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$$\begin{aligned} \mathcal{L}(\delta(t - t_0)) &= \lim_{k \rightarrow 0} \mathcal{L}(d_k(t - t_0)) \\ &= \lim_{k \rightarrow 0} \int_0^{\infty} e^{-st} d_k(t - t_0) dt \\ &= \lim_{k \rightarrow 0} \frac{1}{2k} \int_{t_0 - k}^{t_0 + k} e^{-st} dt \\ &= \lim_{k \rightarrow 0} \frac{-1}{2sk} e^{-st} \Big|_{t_0 - k}^{t_0 + k} \\ &= \lim_{k \rightarrow 0} \frac{1}{2sk} e^{-st_0} (e^{sk} - e^{-sk}) \\ &= \lim_{k \rightarrow 0} \frac{\sinh(sk)}{sk} e^{-st_0} \\ &= \lim_{k \rightarrow 0} \frac{\text{scosh}(sk)}{s} e^{-st_0} = e^{-st_0} \end{aligned}$$

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$

$$\sinh(t) = \frac{e^t - e^{-t}}{2}$$

$$\cosh(t) = \frac{e^t + e^{-t}}{2}$$

$$[\sinh(t)]' =$$

$$[\cosh(t)]' =$$

$$\sinh(0) = \frac{e^0 - e^0}{2} = 0$$

$$\cosh(0) = \frac{e^0 + e^0}{2} = 1$$

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## Intro to Group Theory

Define the  $\cdot$  product on  $R^2$  by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 - y_1x_2)$$

Note  $\cdot$  is

1.) commutative:

$$\begin{aligned} (x_1, y_1) \cdot (x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 - y_1x_2) \\ &= (x_2x_1 - y_2y_1, x_2y_1 - y_2x_1) = (x_2, y_2) \cdot (x_1, y_1) \end{aligned}$$

2.) associative:  $(f \cdot g) \cdot h = f \cdot (g \cdot h)$

3.) distributive w.r.t  $+$ :  $f \cdot (g_1 + g_2) = f \cdot g_1 + f \cdot g_2$

4.)  $(x_1, y_1) \cdot (0, 0) = (0, 0)$

Note  $(0, 1) \cdot (0, 1) = (-1, 0)$

## 6.6: The Convolution Integral

Defn: The convolution of  $f$  and  $g$  is the function  $f * g$  defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t f(x)g(t-x)dx$$

Note  $*$  is

- 1.) commutative:  $f * g = g * f$
- 2.) associative:  $(f * g) * h = f * (g * h)$
- 3.) distributive w.r.t  $+$ :  $f * (g_1 + g_2) = f * g_1 + f * g_2$
- 4.)  $f * 0 = 0 * f = 0$

Example:  $\cos(t) * 1 =$

Example:  $t * t \not\equiv 0$

Thm:  $\mathcal{L}((f * g)(t)) = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t)) = F(s)G(s)$

Proof:

$$\begin{aligned}\mathcal{L}(f(t))\mathcal{L}(g(t)) &= \int_0^\infty e^{-sy} f(y)dy \int_0^\infty e^{-sx} g(x)dx \\ &= \int_0^\infty [\int_0^\infty e^{-sy} f(y)dy] e^{-sx} g(x)dx \\ &= \int_0^\infty [\int_0^\infty e^{-sy} f(y) e^{-sx} g(x)dy] dx \\ &= \int_0^\infty [\int_0^\infty e^{-s(y+x)} f(y)g(x)dy] dx \\ &= \int_0^\infty [\int_0^\infty e^{-s(y+x)} f(y)g(x)dx] dy\end{aligned}$$

Let  $t = x + y$ ,  $dt = dx$

$$\begin{aligned}&= \int_0^\infty [\int_y^\infty e^{-st} f(y)g(t-y)dt] dy \\ &= \int_0^\infty [\int_0^t e^{-st} f(y)g(t-y)dy] dt \\ &= \int_0^\infty e^{-st} [\int_0^t f(y)g(t-y)dy] dt \\ &= \int_0^\infty e^{-st} (f * g)(t) dt \\ &= \mathcal{L}(f * g)\end{aligned}$$

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Cor:  $\mathcal{L}^{-1}(F(s)G(s)) = (f * g)(t) = \int_0^t f(t-s)g(s)ds$

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Example:  $\mathcal{L}^{-1}\left(\frac{1}{s(s-a)}\right) =$